Discrete and Computational Geometry, SS 18
Exercise Sheet "8": VC-dimension/Shatter function
University of Bonn, Department of Computer Science I

- *Written solutions have to be prepared until* **Thursday 5th of July**.
- *You may work in groups of at most two participants.*
- *You can hand over your work to our tutor Raoul Nicolodi in the beginning of the lecture.*

Exercise 25: Applying the epsilon net theorem (4 Points)

Consider the set system $(X, \mathcal{F})$ where $X = [0,1]^2$ is the unit square and $\mathcal{F} = \{X \cap B_{0.1}(x) \mid x \in X\}$. Here, $B_{0.1}(x) := \{y \mid d(x,y) \leq 0.1\}$ is a circle of radius 0.1, centered in x. $d(\cdot,\cdot)$ is the Euclidean distance. The measure $\mu(A)$ of set $A \subset X$ equals the area covered by A.

For any value $0 < \varepsilon \leq 0.01\pi$,

a) Use the *epsilon net Theorem* to obtain an upper bound on the size of an ε net for $(X, \mathcal{F})$. Check the requirements for applying the epsilon net Theorem, i.e. determine the value $\dim_{VC}(\mathcal{F})$ and the value of the constant C as in the proof of the epsilon net Theorem in the lecture.

b) Construct an ε net for $(X, \mathcal{F})$ and compare its size with the value obtained in a).

Exercise 26: Random variables(4 Points)

The *variance* $\mathrm{Var}(X)$ of a random variable X is defined as

$$\mathrm{Var}(X) := \mathrm{E}((X - \mathrm{E}(X))^2)$$

where $\mathrm{E}(\cdot)$ denotes the expected value. Two random variables X, Y, are called *independent*, if for all (measurable) sets, A, B, the equality

$$\mathrm{P}(X \in A \ \wedge \ Y \in B) = \mathrm{P}(X \in A) \cdot \mathrm{P}(Y \in B)$$

is fulfilled. They are called *uncorrelated*, if

$$\mathrm{E}(X \cdot Y) = \mathrm{E}(X) \cdot \mathrm{E}(Y)$$

holds.

a) Give a simple example of two random variables which are independent, but not uncorrelated.

b) Show that if X and Y are independent random variables, which attain finitely many values only, then X and Y are also uncorrelated.

c) Prove that if X and Y are two uncorrelated random variables, then $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ holds.

Exercise 27: Packings and transversals (4 Points)

Let natural numbers $k \le n$ be given. We consider the basic set $X = \{1, \dots, n\}$ and the set system

$$\mathcal{F} := \{Y \subseteq X \mid |Y| = k\}.$$

A subset $T \subseteq X$ is called a *transversal* of $\mathcal{F}$ if it intersects all the (non-empty) sets of $\mathcal{F}$. The *transversal number*, denoted by $\tau(\mathcal{F})$, is the smallest possible cardinality of a transversal of $\mathcal{F}$. The *packing number* of $\mathcal{F}$, denoted by $\nu(\mathcal{F})$, is the maximum cardinality of a system of pairwise disjoint sets in $\mathcal{F}$.

$$\nu(\mathcal{F}) = \sup\{|M| : M \subseteq \mathcal{F}, M_1 \cap M_2 = \emptyset \text{ for all } M_1, M_2 \in M, M_1 \neq M_2\}$$

For a finite set X, as in this exercise, we define a *fractional transversal* for $\mathcal{F}$ to be a function $\phi : X \mapsto [0,1]$ such that for each $S \in \mathcal{F}$, we have $\sum_{x \in S} \phi(x) \ge 1$. The size of a fractional transversal ϕ is $\sum_{x \in X} \phi(x)$, and the *fractional transversal number* $\tau^*(\mathcal{F})$ is the infimum of the sizes of fractional transversals. A *fractional packing* for $\mathcal{F}$ is a function $\psi : \mathcal{F} \mapsto [0,1]$ where for each $x \in X$, we have $\sum_{S \in F : x \in S} \psi(S) \le 1$. The *size* of a fractional packing ψ is $\sum_{S \in F} \psi(S)$, and the *fractional packing number* $\nu^*(\mathcal{F})$ is the supremum of the sizes of all fractional packings for $\mathcal{F}$.

For the given base set X and set system $\mathcal{F}$, determine the transversal number $\tau(\mathcal{F})$, the packing number $\nu(\mathcal{F})$, and their fractional variants $\tau^*(\mathcal{F})$ and $\nu^*(\mathcal{F})$.