Online Motion Planning MA-INF 1314
Searching Points/Rays

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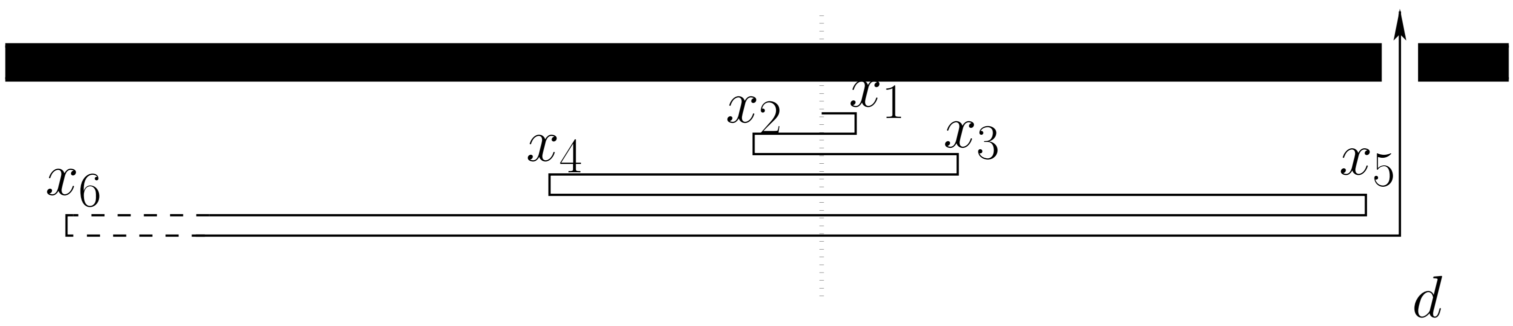
Rep.: Searaching for a point!

- 2-ray search: Point on a line
- Compare with shortest path, competitive?
- Reasonable strategy: Depth x_1, depth x_2 and so on
- Traget at least step 1 away!
- Worst-Case, just behind d, one add. turn!
- Strategy, such that: $\sum_{i=0}^{k+1} 2x_i + x_k \leq C x_k$
- Minimize: $\frac{\sum_{i=0}^{k+1} x_i}{x_k}$, Functional!



Rep.: Theorem Gal 1980

If functional F_k fulfils:

i) F_k continuous

ii) F_k unimodal: $F_k(A \cdot X) = F_k(X)$ und $F_k(X+Y) \leq \max\{F_k(X), F_k(Y)\}$,

iii) $\liminf_{a \mapsto \infty} F_k\left(\frac{1}{a^{k+i}}, \frac{1}{a^{k+i-1}}, \ldots, \frac{1}{a}, 1\right) = \liminf_{\epsilon_{k+i}, \epsilon_{k+i-1}, \ldots,, \epsilon_1 \mapsto 0} F_k\left(\epsilon_{k+i}, \epsilon_{k+i-1}, \ldots, \epsilon_1, 1\right)$,

iv) $\liminf_{a \mapsto 0} F_k\left(1, a, a^2, \ldots, a^{k+i}\right) = \liminf_{\epsilon_{k+i}, \epsilon_{k+i-1}, \ldots \epsilon_1 \mapsto 0} F_k\left(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{k+i}\right)$,

v) $F_{k+1}(f_1, \ldots, f_{k+i+1}) \geq F_k(f_2, \ldots, f_{k+i+1})$.

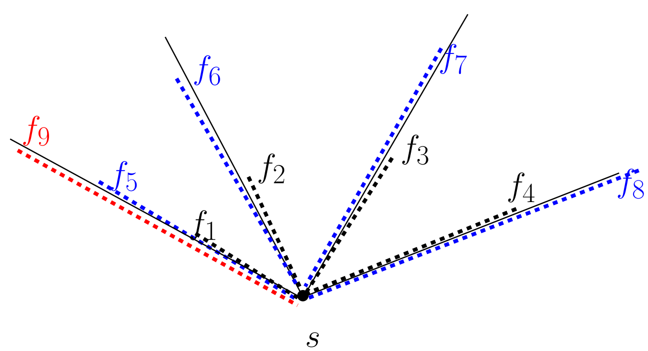
Then:$\sup_k F_k(X) \geq \inf_a \sup_k F_k(A_a)$ mit $A_a = a^0, a^1, a^2, \ldots$ und $a > 1$.

Rep.: Example 2-ray search

- $F_k(f_1, f_2, \ldots) := \frac{\sum_{i=1}^{k+1} f_i}{f_k}$ for all k.
- Unimodal $F_k(A \cdot X) = F_k(X)$ and $F_k(X + Y) \leq \max\{F_k(X), F_k(Y)\}$?
- $\frac{\sum_{i=1}^{k+1} A \cdot f_i}{A \cdot f_k} = \frac{\sum_{i=1}^{k+1} f_i}{f_k}$
- $F_k(X + Y) \leq \max\{F_k(X), F_k(Y)\}$?
- Follows from $\frac{a}{b} \geq \frac{c}{d} \Leftrightarrow \frac{a+c}{d+b} \leq \frac{a}{b}$
- Simple equivalence!
- Optimize: $f_k(a) := \frac{\sum_{i=1}^{k+1} a^i}{a^k}$
- Minimized by $a = 2$

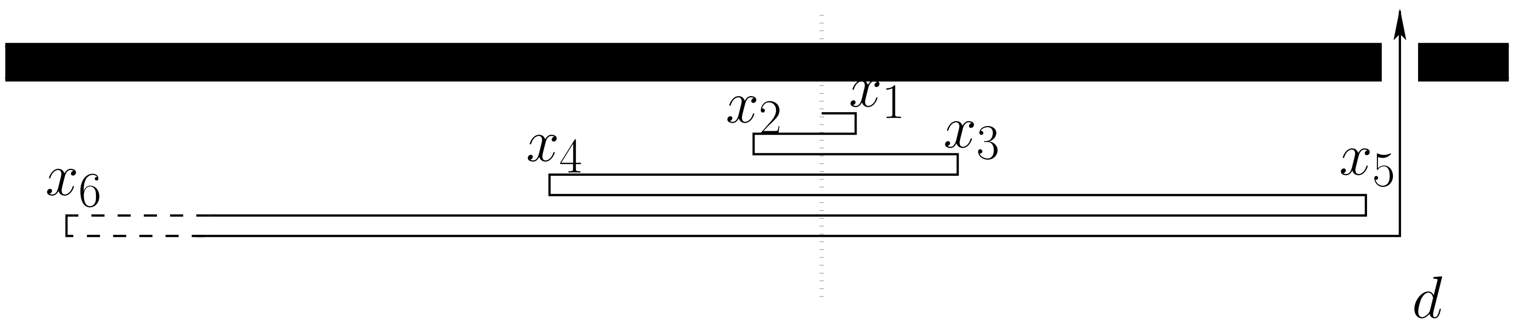
Rep.: Search on m-rays

- **Lemma** For the m-ray search problem there is always an optimal competitive strategy $(f_1, f_2, \ldots)$ that visits the rays in a periodic order and with overall increasing depth.
- periodic and monotone: (f_j, J_j), $J_j = j + m$, $f_j \geq f_{j-1}$
- Proof: First index with: $f_j > f_{j+1}$, $J_j > J_{j+1}$, Exchange values and the order on the rays, successively!
- (f_j, J_j), $J_j = j + m$, $f_j \geq f_{j-1}$ Theorem of Gal can be applied!



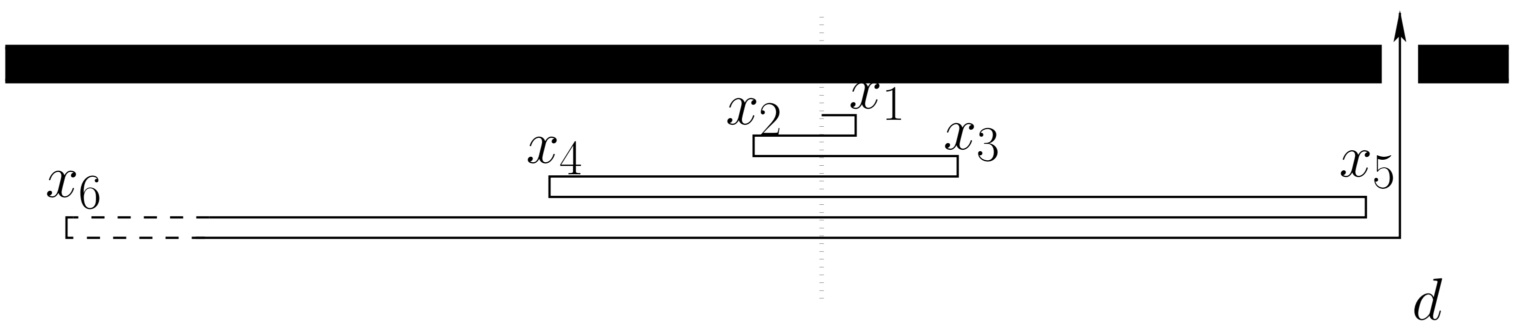
Other approach: Optimality for equations!

- Reasonable strategy, ratio: $\frac{\sum_{i=1}^{k+1} 2x_i + x_k}{x_k} = 1 + 2\frac{\sum_{i=1}^{k+1} x_i}{x_k}$
- Ass.: C optimal, $\frac{\sum_{i=1}^{k+1} x_i}{x_k} \leq \frac{(C-1)}{2}$
- There is strategy $(x'_1, x'_2, x'_3 \ldots)$ s. th. $\frac{\sum_{i=1}^{k+1} x'_i}{x'_k} = \frac{(C-1)}{2}$ for all k
- Monotonically increasing in x'_j $(j \neq k)$, decreasing in x'_k
- First k with: $\frac{\sum_{i=1}^{k+1} x_i}{x_k} < \frac{(C-1)}{2}$, decrease x_k
- $\frac{\sum_{i=1}^{k} x_i}{x_{k-1}} < \frac{(C-1)}{2}$!, x_{k-1} decrease etc., monotonically decreasing sequence, bounded, converges! Non-constructive!



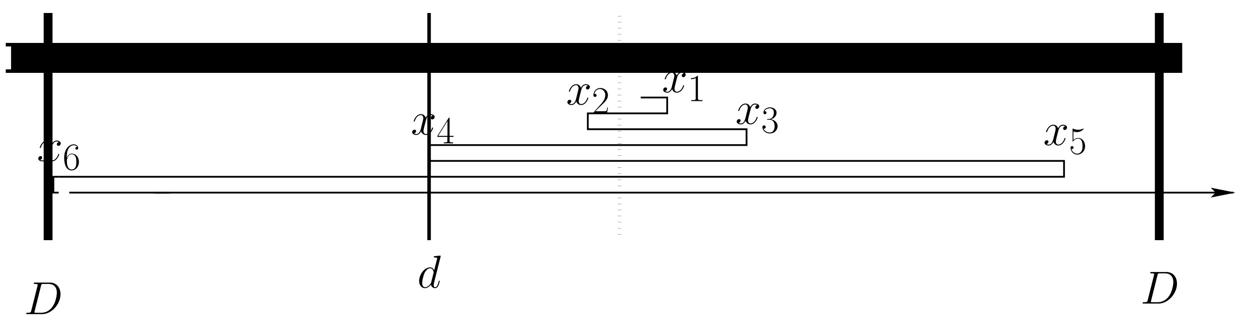
Other approach: Optimality for equations!

- Set: $\frac{\sum_{i=1}^{k+1} x'_i}{x'_k} = \frac{(C-1)}{2}$ for all k
- $\sum_{i=1}^{k+1} x'_i - \sum_{i=1}^{k} x'_i = \frac{(C-1)}{2}\left(x'_k - x'_{k-1}\right)$
- Thus: $C'\left(x'_k - x'_{k-1}\right) = x'_{k+1}$, Recurrence!
- Solve a recurrence! Analytically! Blackboard!
- Characteristical polynom: No solution $C' < 4$
- $x'_i = (i+1)2^i$ with $C' = 4$ is a solution! Blackboard! Optimal!



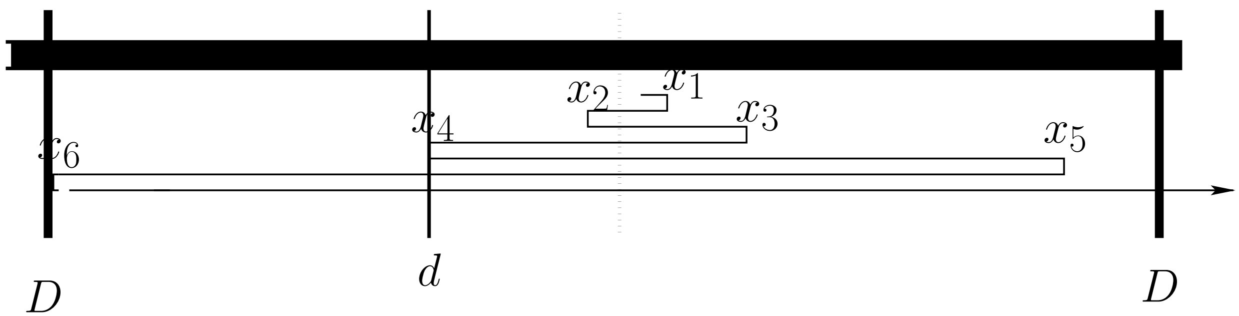
2-ray search, restricted distance

- Assume goal is no more than dist. $\leq D$ away
- Exactly D! Simple ratio 3!
- Find optimal startegy, minimize C!
- Vice-versa: C is given! Find the largest distance D (reach R) that still allows C competitive search.
- One side with $f_{\text{Ende}} = R$, the other side arbitrarily large!



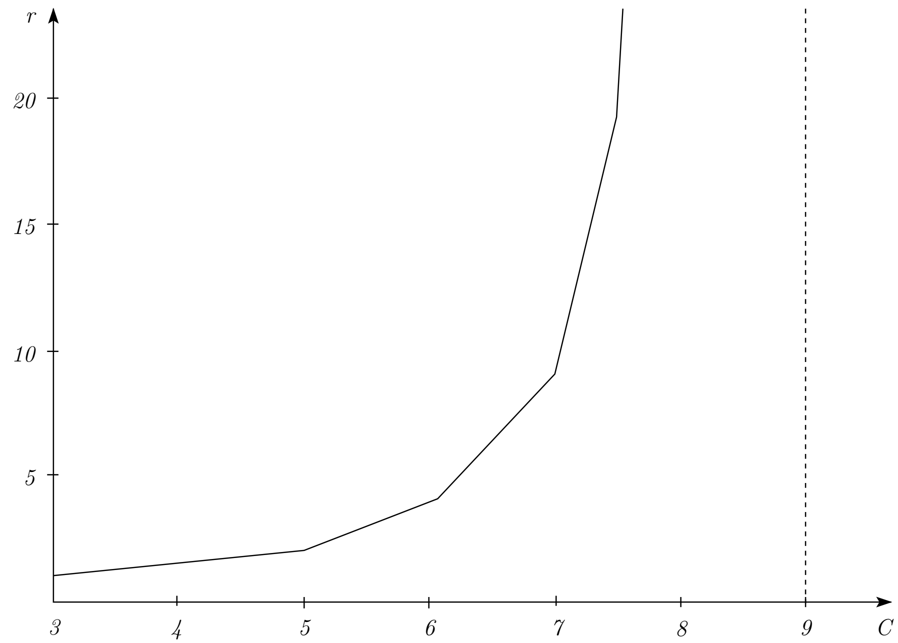
2-ray search, maximal reach R

- C given, optimal reach R!
- **Theorem** The strategy with equality in any step maximizes the reach R !
- Strategy: $\frac{\sum_{i=1}^{k+1} x_i}{x_k} = \frac{(C-1)}{2}$, first step: $x_1 = \frac{(C-1)}{2}$
- Recurrence: $x_0 = 1$, $x_{-1} = 0$, $x_{k+1} = \frac{(C-1)}{2}(x_k - x_{k-1})$
- Strategy is optimal! By means of the Comp. Geom. lecture!



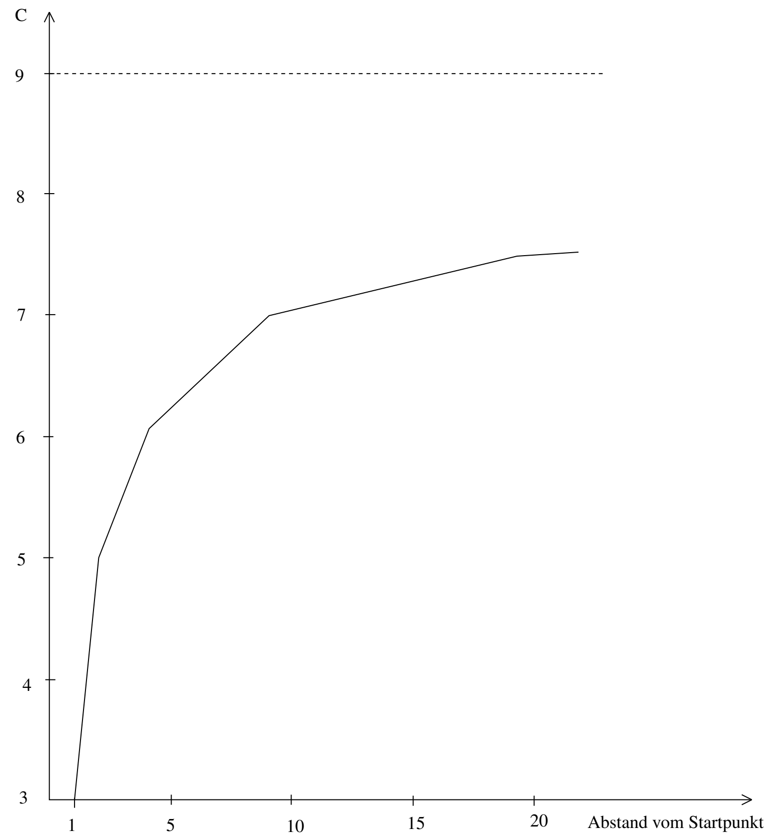
2-ray search, maximal reach R

- $f(C) :=$ maximal reach depending on C
- Bends are more steps!



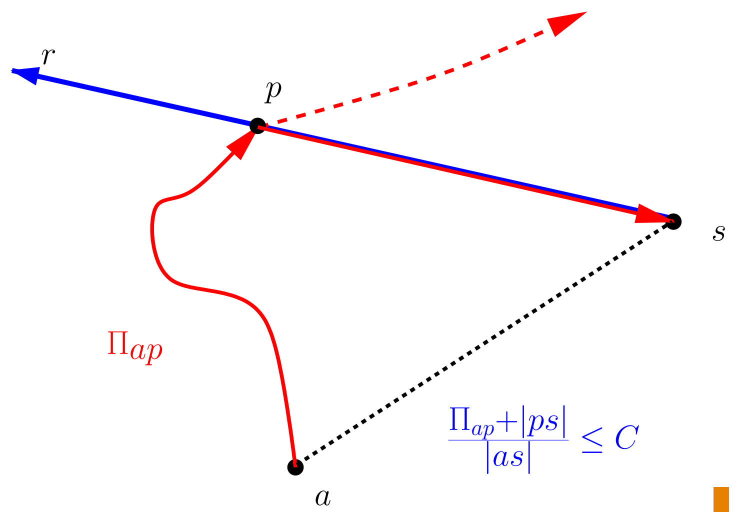
2-ray search, given distance R

- $f(C) :=$ maximal reach depending on C
- Rotate, R given, binary search!



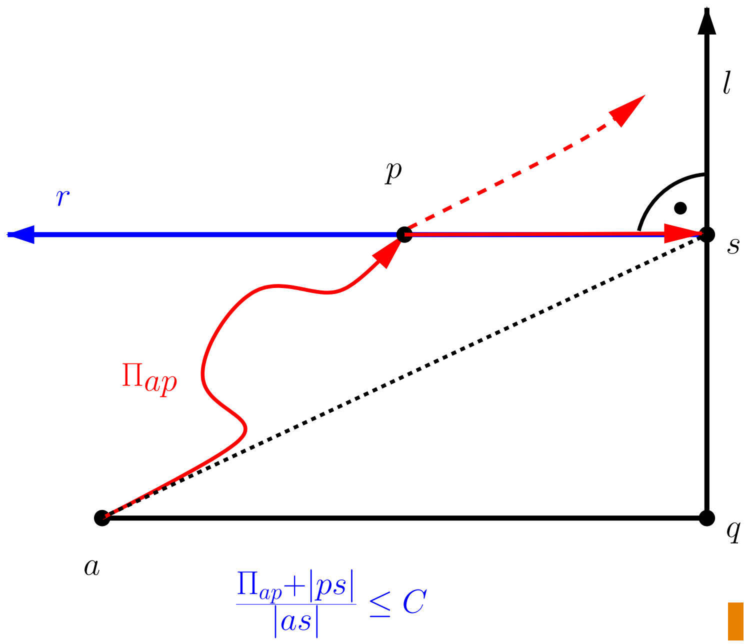
Searching for the origin of ray

- Unknown ray r in the plane, unknown origin s
- Startpoint a
- Searchpath Π, hits r, detects s, move to s
- Shortest path OPT, build the ratio
- Π has *competitive ratio* C if inequality holds for all rays
- Task: Find searchpath Π with the minimal C



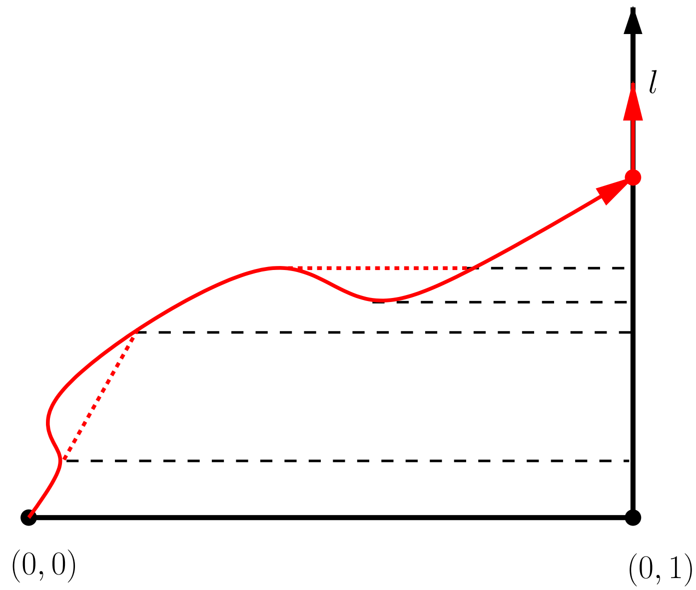
The Window-Shopper-Problem

- Unknown ray starts at s on *known* vertical line l(window)
- Ray starts perpendicular to l
- aq runs parallel to r
- *Motivation:* Move along a window until you *detect* an item
- Move to the item



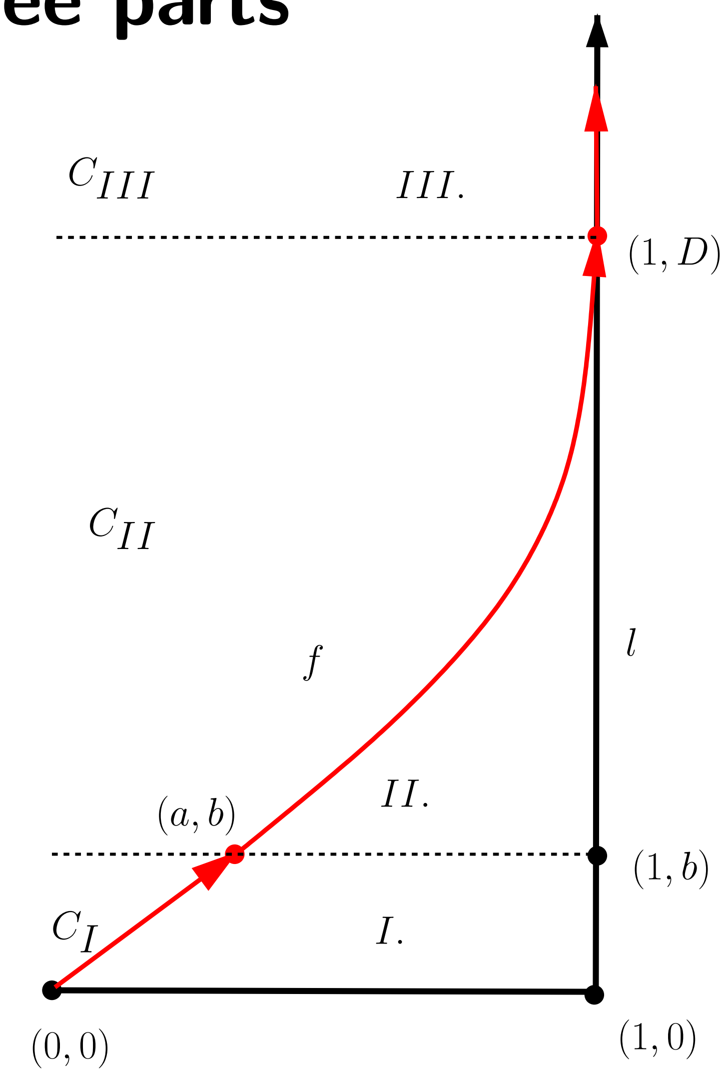
Some observations

- Any reasonable strategy is monotone in x and y
- Otherwise: Optimize for some s on l
- Finally hits the *window*
- Ratio is close to 1 in the beginning, but bigger than 1
- Ratio goes to 1 at the end



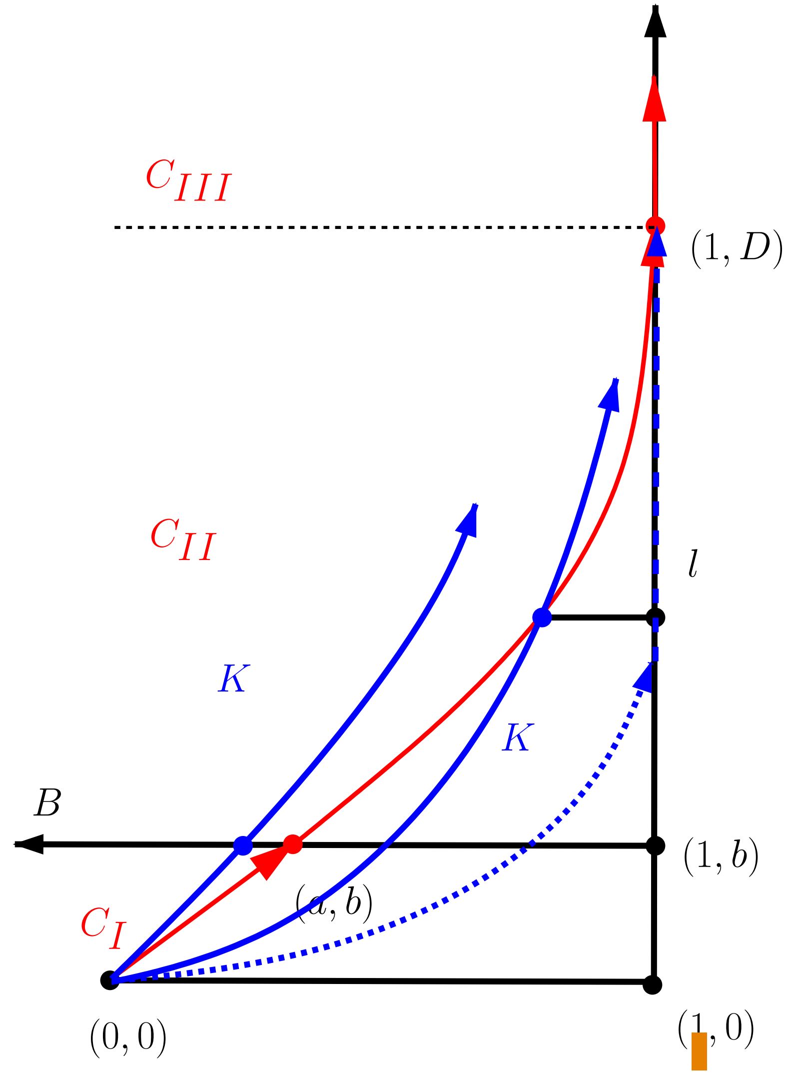
Strategy design: Three parts

- A line segment from $(0,0)$ to (a,b) with increasing ratio for s between $(1,0)$ and $(1,b)$
- A curve f from (a,b) to some point $(1,D)$ on l which has the same ratio for s between $(1,b)$ and $(1,D)$
- A ray along the *window* starting at $(1,D)$ with decreasing ratio for s beyond $(1,D)$ to infinity
- Worst-case ratio is attained for all s between $(1,b)$ and $(1,D)$



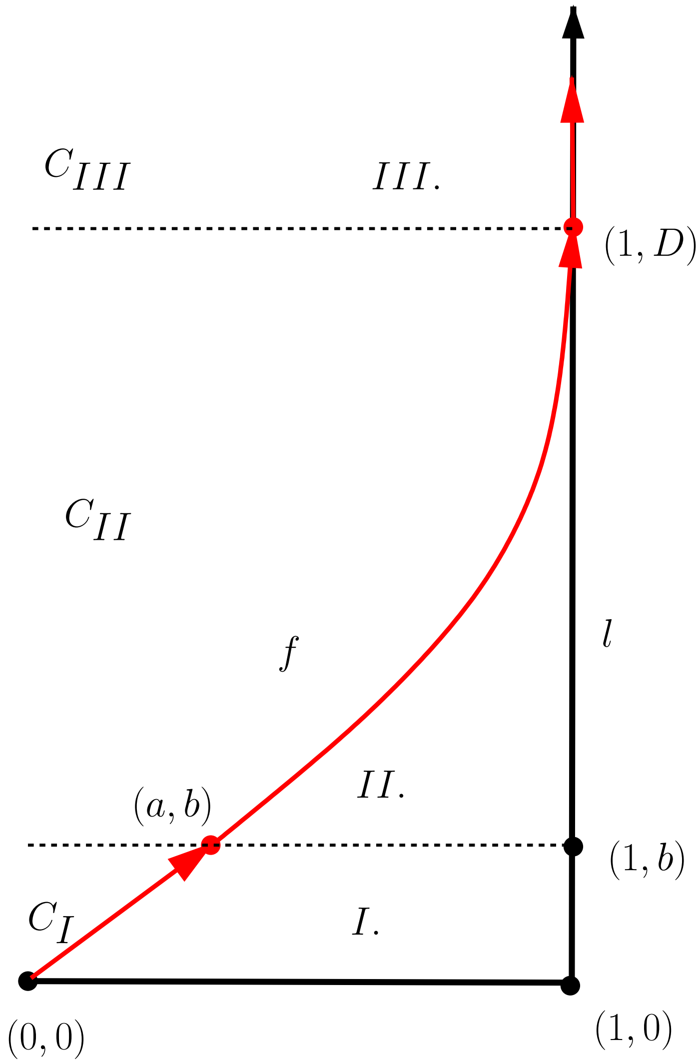
Optimality of this strategy

- By construction
- Curve has the given property
- Proof: Curve is convex
- Assume: Optimal curve K
- K hits ray B at some point (x, b)
- Two cases:
  - Hits B to the left of a: ratio is bigger
  - Cross f beyond B from the right: ratio is bigger



Design of the strategy: By conditions

- 1) Monotonically increasing ratio for s from $(1,0)$ to $(1,b)$
- 2) Constant ratio for s from $(1,b)$ to $(1,D)$
- Determines a, b and D



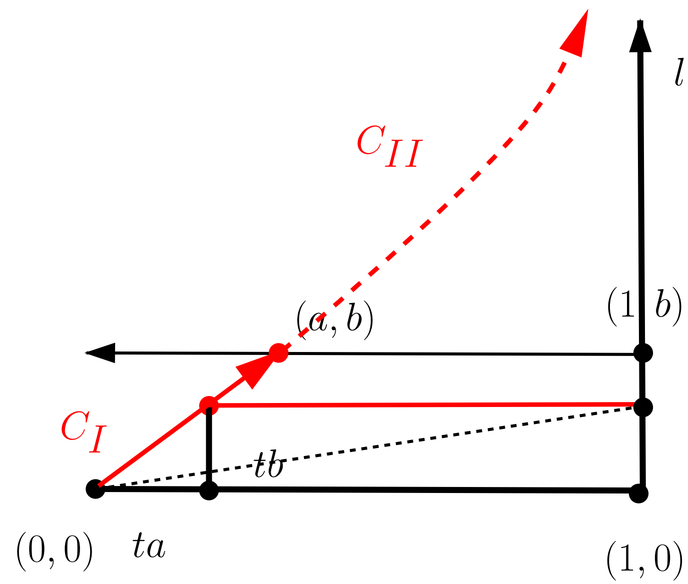
Design of the strategy: Condition 1)

- Start with 1): Ratio for $t \in [0,1]$:

$$\phi(t) = \frac{t\sqrt{a^2+b^2}+1-ta}{\sqrt{1+t^2b^2}}$$

- Monotonicity: $\phi'(t) \geq 0 \qquad \forall t \in [0,1]$
- Analysis:
$\Leftrightarrow \sqrt{a^2+b^2} - a \geq tb^2 \qquad \forall t \in [0,1]$
- $\Leftrightarrow b^2 \leq 1 - 2a$
- Choose: $a = \frac{1-b^2}{2}$
- Worst-case ratio:

$$C = \frac{\sqrt{a^2+b^2}+1-a}{\sqrt{1+b^2}} = \sqrt{1+b^2}$$



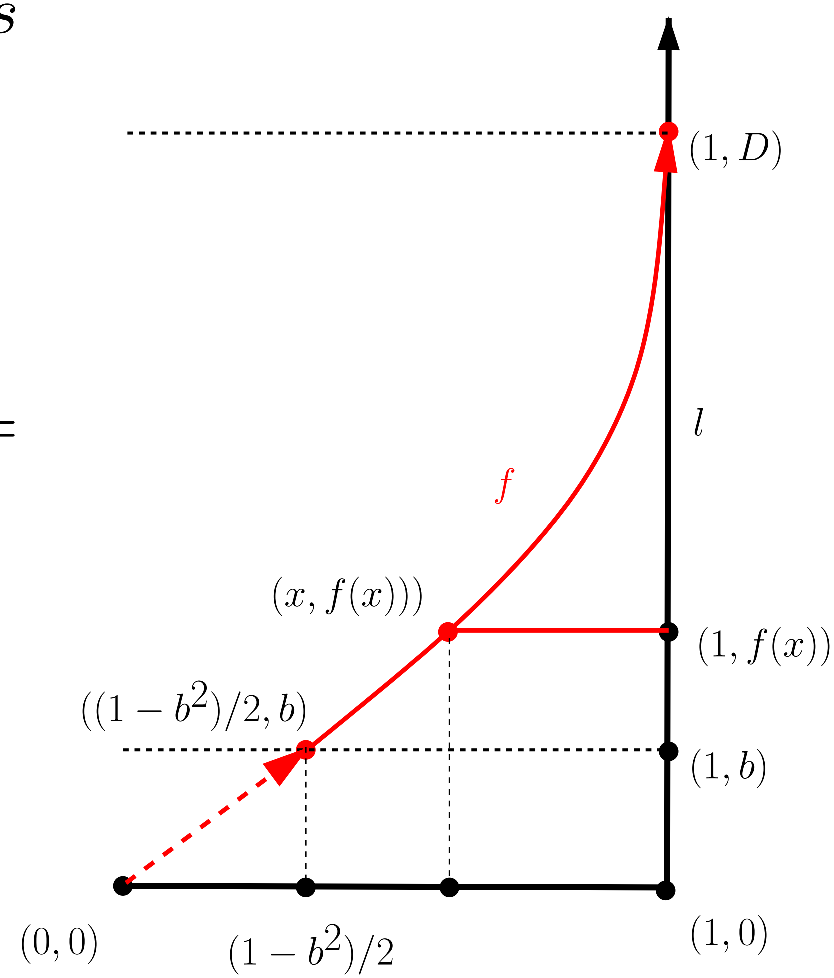
Design of the strategy: Condition 2)

- 2) Constant ratio $C = \sqrt{1+b^2}$ for s from $(1,b)$ to $(1,D)$
- Function $f(x)$ for $x \in [a,1]$
- Constant ratio C:

$$\sqrt{a^2+b^2} + \int_a^x \sqrt{1+f'(t)^2}dt + 1 - x = C \cdot \sqrt{1+f(x)^2}$$

- Transformations ($f'(x) \neq 0!$):

$$\Leftrightarrow f'(x) = 2C\frac{\sqrt{1+f(x)^2}f(x)}{1+(1-C^2)f(x)^2}$$



Solutions for $y = f(x)$

- $f'(x) = 2\sqrt{1+b^2}\frac{\sqrt{1+f(x)^2}f(x)}{1-b^2f(x)^2}$, $((1-b^2)/2, b)$ on the curve
- Solve: $y' = 1 \cdot 2\sqrt{1+b^2}\frac{\sqrt{1+y^2}y}{1-b^2y^2}$ for y with $((1-b^2)/2, b)$
- First order diff. eq. $y' = h(x)g(y)$, separated variables, point (k, l)
- Solution: $\int_l^y \frac{dt}{g(t)} = \int_k^x h(z)dz$

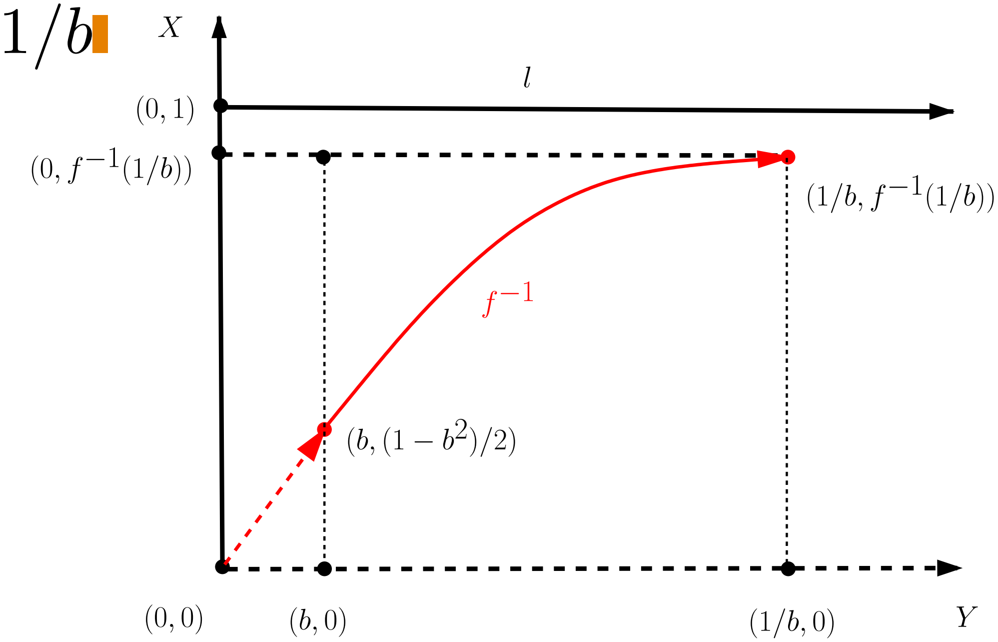
$$\int_b^y \frac{1-b^2t^2}{2\sqrt{1+b^2}\sqrt{1+t^2}t}dt = \int_{(1-b^2)/2}^x 1 \cdot dz = x - (1-b^2)/2$$

$$x = -\frac{b^2\sqrt{1+y^2} + \text{arctanh}\left(1/(\sqrt{1+y^2})\right) - \text{arctanh}\left(1/(\sqrt{1+b^2})\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}$$

- Solution for inverse function $x = f^{-1}(y)$, for $y \in [b, 1/b]$

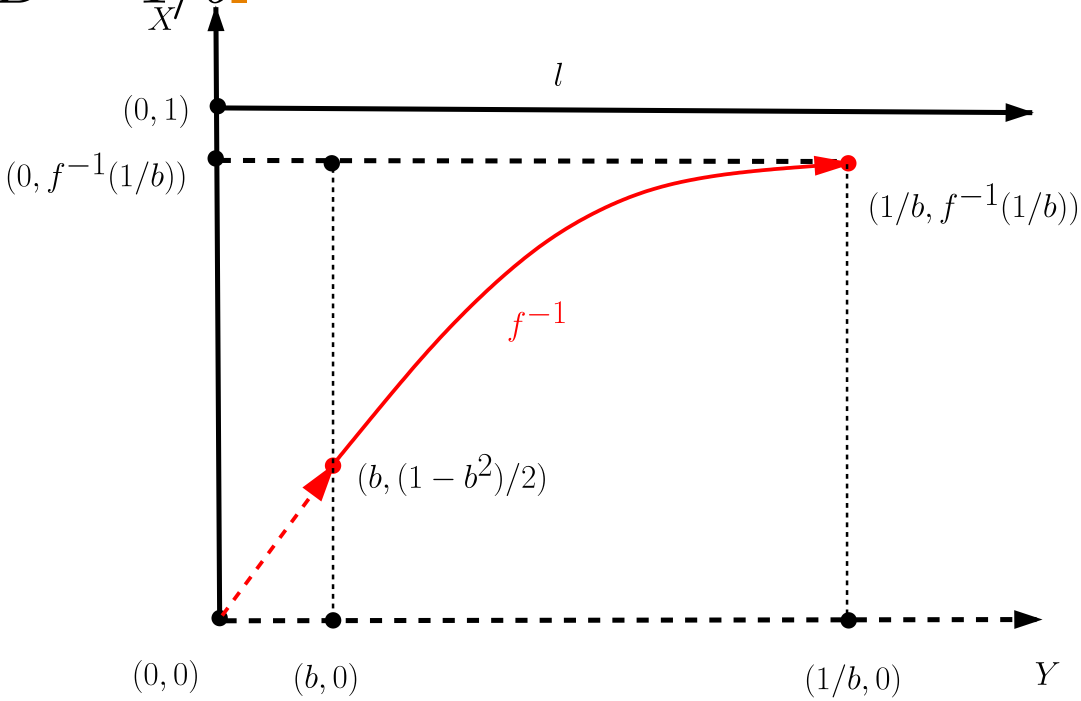
Consider inverse function $x = f^{-1}(y)$

- $x' = \frac{1}{g(y)} = -\frac{(b^2y^2-1)}{2\sqrt{1+y^2}y\sqrt{(1+b^2)}} \geq 0$ for $y \in [b, 1/b]$
- $x'' = -\frac{(b^2y^2+2y^2+1)}{2(1+y^2)^{3/2}\sqrt{1+b^2y^2}} \leq 0$ for $y \geq 0$
- $x = f^{-1}(y)$ concave, $y = f(x)$ convex
- Max. at $y = 1/b$



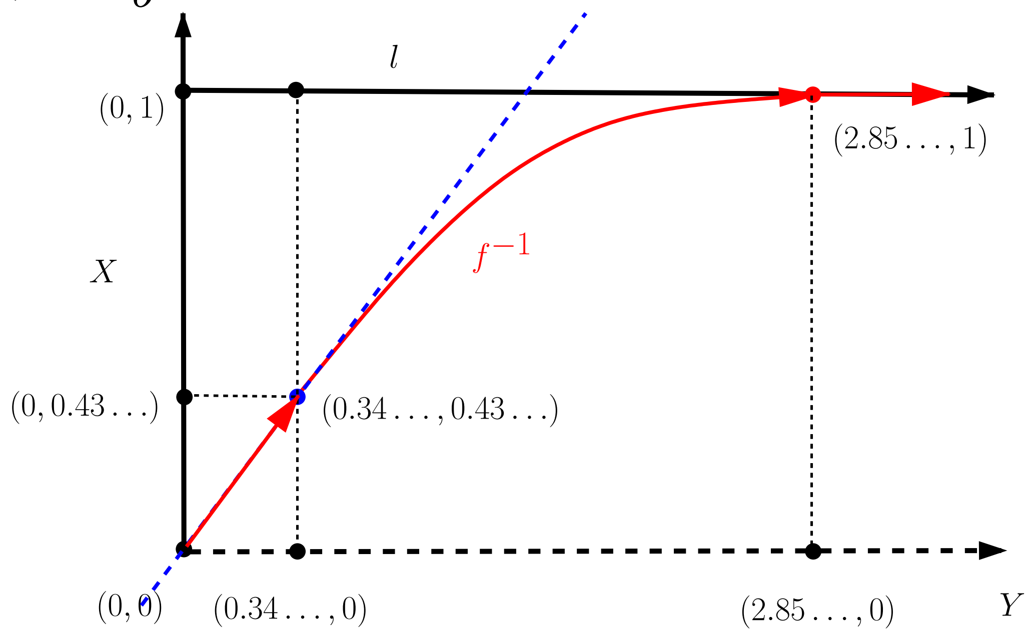
Consider inverse function $x = f^{-1}(y)$

- Maximum at $y = 1/b$
- Find b so that $f^{-1}(1/b) = 1$
- Fixes b and $D = 1/b$



Optimality of f (or f^{-1})

- Solve $f^{-1}(1/b) = 1$: $b = 0.3497\ldots$, $D = 1/b = 2.859\ldots$, $a = 0.43\ldots$, worst-case ratio $C = \sqrt{1 + b^2} = 1.05948\ldots$
- f convex from (a, b) to $(1, D)$, line segment convex
- Prolongation of line segment is tangent of f^{-1} at (b, a)
- Insert: $f^{-1\prime}(b) = \frac{a}{b}$!



Conclusion

- Optimal strategy with ratio
  $C = 1.05948\ldots$