Online Motion Planning MA-INF 1314
Alternative cost measures!

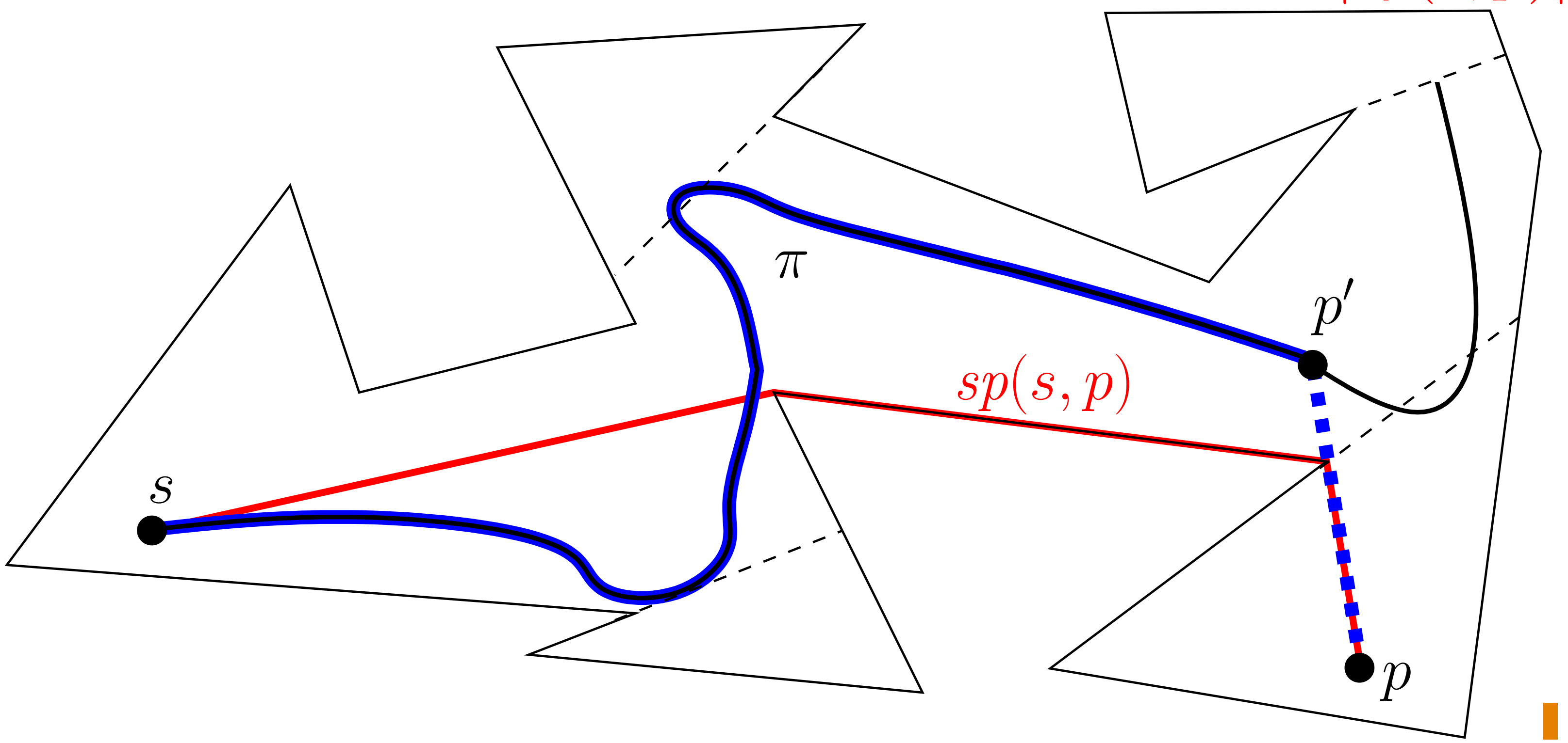
Elmar Langetepe
University of Bonn

Rep.: Optimal searchpath

- We have seen:
  Searching for a goal (polygon) in general not competitive
- Question: What is a good searchpath (for polygons)?
- Searching: Target point unknown!
- Offline-Searching: Environment is known
- Online-Searching: Environment unknown

Rep.: Search ratio for polygons

π: **Searchpath**, quality for π: $\mathrm{SR}(\pi, P) = \max_{p \in P} \frac{|\pi_s^{p'}| + |p'p|}{|\mathrm{sp}(s,p)|}$



Rep.: Quality measures!

- *Competitive ratio* of search strategy $\mathcal{A}$ in polygons:

$$C := \sup_{P} \sup_{p \in P} \frac{|\mathcal{A}(s,p)|}{|\mathsf{sp}(s,p)|}$$

Rep.: Search ratio in general

Given: Environment $\mathcal{E}$, Set of goals $\mathcal{G} \subseteq \mathcal{E}$

Graphs $G = (V, E)$: **Vertices** $\mathcal{G} = V$

Geometric Search $\mathcal{G} = V \cup E$

(Requirement: $\forall p \in \mathcal{E} : |\mathsf{sp}(s,p)| = |\mathsf{sp}(p,s)|$)

Search ratio of a search strategy $\mathcal{A}$ for $\mathcal{E}$:

$$\mathsf{SR}(\mathcal{A}, \mathcal{E}) := \sup_{p \in \mathcal{G}} \frac{|\mathcal{A}(s,p)|}{|\mathsf{sp}(s,p)|}$$

Optimal search ratio:

$$\mathsf{SR}_{\mathsf{OPT}}(\mathcal{E}) := \inf_{\mathcal{A}} \mathsf{SR}(\mathcal{A}, \mathcal{E})$$

Rep.: Search ratio approximation

- *Competitive ratio* : $C := \sup_{\mathcal{E}} \sup_{p \in \mathcal{G}} \frac{|\mathcal{A}(s,p)|}{|\mathsf{sp}(s,p)|}$

- *Search ratio:* $\mathsf{SR}(\mathcal{A},\mathcal{E}) := \sup_{p \in \mathcal{G}} \frac{|\mathcal{A}(s,p)|}{|\mathsf{sp}(s,p)|}$

- *Optimal search ratio*: $\mathsf{SR}_{\mathsf{OPT}}(\mathcal{E}) := \inf_{\mathcal{A}} \mathsf{SR}(\mathcal{A},\mathcal{E})$

- Approximation: $\mathcal{A}$ *search-competitiv*

$$C_s := \sup_{\mathcal{E}} \frac{\mathsf{SR}(\mathcal{A},\mathcal{E})}{\mathsf{SR}_{\mathsf{OPT}}(\mathcal{E})}$$

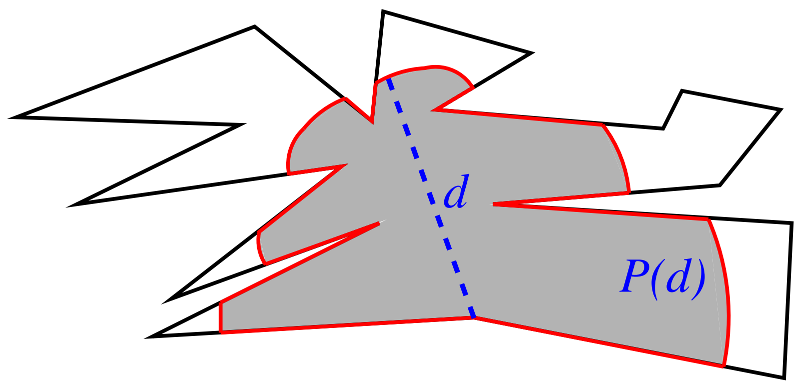
- Comparison not against SP, but against best possible SR

Rep.: Depth-restricted exploration

Def. Exploration-Strategy Expl for $\mathcal{E}$ is called **depth-restrictable**, if we can derive a strategy $\mathrm{Expl}\,(d)$ such that:

- $\mathrm{Expl}\,(d)$ explores $\mathcal{E}$ up to depth $d \geq 1$
- return to s after the exploration
- $\mathrm{Expl}\,(d)$ is C-competitiv, i.e.,$\exists\, C \geq 1 : \forall \mathcal{E}$:

$$|\mathrm{Expl}\,(d)| \leq C \cdot |\mathrm{Expl}_{\mathrm{OPT}}(d)|\,.$$



Rep.: Searchpath approximation

Algorithm

- Explore $\mathcal{E}$ by increasing depth: $\mathrm{Expl}\,(2^i)$ für $i = 1, 2, \ldots$

Lemma:

- Roboter without vision system
- Environment $\mathcal{E}$
- $\mathrm{Expl}_{\mathrm{ONL}}$: C-competitive, depth-restrictable, online exploration strategy for $\mathcal{E}$
  (d. h. $|\mathrm{Expl}\,(d)| \leq C \cdot |\mathrm{Expl}_{\mathrm{OPT}}(d)|$)

$\Rightarrow$ Algorithm gives $4C$-Approximation of optimal search path!

Rep.: Searchpath approximation proof

$$|\Pi_{\mathrm{Expl}_{\mathrm{opt}}(d)}| \leq d \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \tag{1}$$

$$\begin{aligned}
\mathsf{SR}(\Pi) &\leq \frac{\sum_{i=1}^{j+1} |\Pi_{\mathrm{Expl}(2^i)}|}{2^j + \varepsilon} \\
&\underset{(\mathsf{Ass.})}{\leq} \frac{C}{2^j} \sum_{i=1}^{j+1} |\Pi_{\mathrm{Expl}_{\mathrm{opt}}(2^i)}| \underset{(1)}{\leq} \frac{C}{2^j} \sum_{i=1}^{j+1} 2^i \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \\
&\leq C \cdot \left(\frac{2^{j+2} - 1}{2^j} \right) \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \leq 4C \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1)
\end{aligned}$$

Rep.: Applications

$\Rightarrow$ Searchpath approximation of factor 4

- Graphs: Online and Offline!
  CFS ($C = 4 + \frac{8}{\alpha}$) depth-restrictable!!
- But: Factor depends on rope length $(1+\alpha)r$ by depth r
- CFS sometimes explores more than d (precisely $(1+\alpha)d$)

$\Rightarrow$ $\mathrm{Expl}\,(d)$ not comparable to $\mathrm{Expl}_{\mathrm{OPT}}(d)$

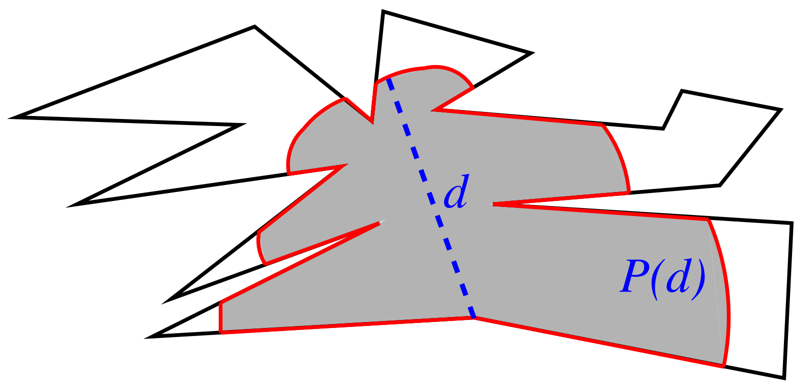
- Workaround: Compare $\mathrm{Expl}\,(d)$ with $\mathrm{Expl}_{\mathrm{OPT}}(\beta \cdot d)$

β-depth restricted exploration

Def. Exploration strategy Expl for $\mathcal{E}$ is denoted as **β-depth restrictable**, if we can derive a strategy $\mathrm{Expl}\,(d)$ such that:

- $\mathrm{Expl}\,(d)$ explores $\mathcal{E}$ only up to depth $d \geq 1$
- returns to the start s
- $\mathrm{Expl}\,(d)$ is C_β-competitiv, i.e., $\exists\, C_\beta \geq 1, \beta > 0 :\ \forall \mathcal{E}$:

$$|\mathrm{Expl}\,(d)| \leq C_\beta \cdot |\mathrm{Expl}_{\mathrm{OPT}}(\beta \cdot d)|\,.$$



Searchpath approximation

Theorem:

- Agent without vision
- Environment $\mathcal{E}$
- $\mathrm{Expl}:$ C_β-competitive, β-depth restrictable, online exploration strategy for $\mathcal{E}$, (i.e., $|\mathrm{Expl}\,(d)| \leq C_\beta \cdot |\mathrm{Expl}_{\mathrm{OPT}}(\beta \cdot d)|$)

$\Rightarrow$ Algorithm (exploration/double depth) gives a $4\beta C_\beta$-approximation of the optimal searchpath

Corollary: Unknown graphs, Algorithm with CFS is $4(1+\alpha)(4+\frac{8}{\alpha})$-approximation of optimal searchpath

Searchpath Approximation Proof

$$|\Pi_{\mathrm{Expl}_{\mathrm{opt}}(d)}| \leq d \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \tag{2}$$

$$\begin{aligned}
\mathsf{SR}(\Pi) &\leq \frac{\sum_{i=1}^{j+1} |\Pi_{\mathrm{Expl}(2^i)}|}{2^j + \varepsilon} \\
&\leq \frac{C_\beta}{2^j} \sum_{i=1}^{j+1} |\Pi_{\mathrm{Expl}_{\mathrm{opt}}(\beta 2^i)}| \underset{(2)}{\leq} \frac{C_\beta}{2^j} \sum_{i=1}^{j+1} \beta 2^i \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \\
&\leq C_\beta \beta \cdot \left(\frac{2^{j+2} - 1}{2^j} \right) \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1) \leq 4\beta C_\beta \cdot (\mathsf{SR}(\Pi_{\mathrm{opt}}) + 1)
\end{aligned}$$

Non approximation results: Theorem

No constant approximation of the search ratio!

1. Planar graph $G = (V, E)$ multiple edges, goal set V.
2. General graph $G = (V, E)$ goal set V.
3. Directed graph $G = (V, E)$ goal set E and V.

Counter examples, lower bound! Blackboard!

Searching with vision!

Problem: Return path from $\mathrm{last}(d)$ to s has length $\leq d$, might be false! But: $\mathrm{sp}(\mathrm{last}(d), s) \leq |\pi_{\mathrm{OPT}s}^{\mathrm{last}(d)}|$

Robot still has to move to t

Theorem:

- Roboter with vision
- Environment $\mathcal{E}$
- $\mathrm{Expl}:$ C_β-competitive, β-depth restrictable, Online Explorationstrategy for $\mathcal{E}$ (i.e. $|\mathrm{Expl}\,(d)| \leq C_\beta \cdot |\mathrm{Expl}_{\mathrm{OPT}}(\beta \cdot d)|$)

$\Rightarrow$ Algorithm gives $8\beta C_\beta$-Approximation of optimal search ratio.

Proof of the Theorem

$$\mathsf{SR}(\Pi_{\text{opt}}) \geq \frac{\left|\pi_{\text{OPT}s}{}^{\text{last}(d)}\right|}{d} \geq \frac{\left|\Pi_{\text{Expl}_{\text{opt}}(d)}\right|}{2d} \Leftrightarrow \left|\Pi_{\text{Expl}_{\text{opt}}(d)}\right| \leq 2d{\cdot}\mathsf{SR}(\Pi_{\text{opt}})$$

Ratio against search path:

$$\begin{aligned} \frac{\sum_{i=1}^{j+1} \left|\Pi_{\text{Expl}_{\text{onl}}(2^i)}\right|}{2^j} &\leq C_\beta \cdot \frac{\sum_{i=1}^{j+1} \left|\Pi_{\text{Expl}_{\text{opt}}(\beta 2^i)}\right|}{2^j} \leq 2C_\beta \cdot \frac{\sum_{i=1}^{j+1} \beta 2^i \, \mathsf{SR}(\Pi_{\text{opt}})}{2^j} \\ &\leq 8\beta C_\beta \cdot \mathsf{SR}(\Pi_{\text{opt}}) \,. \end{aligned}$$

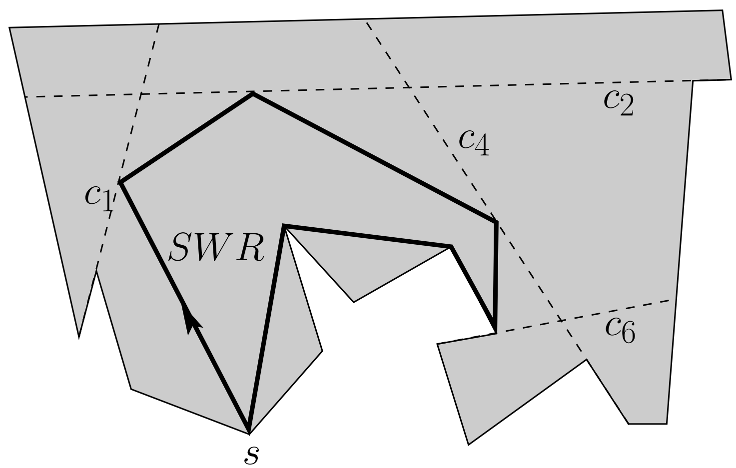
Outlook: Applications!

- Simple polygon, Offline: SWR ($C_\beta = 1 = \beta$)
  $\Rightarrow$ 8-Approximation
- Rectilinear Polygons, Online: Greedy-Online ($C_\beta = \sqrt{2}, \beta = 1$)
  $\Rightarrow 8\sqrt{2}$-Approximation
- Simple Polygons, Online: PolyExplore ($C_\beta = 26, \beta = 1$)
  $\Rightarrow$ 212-Approximation

Exploration!

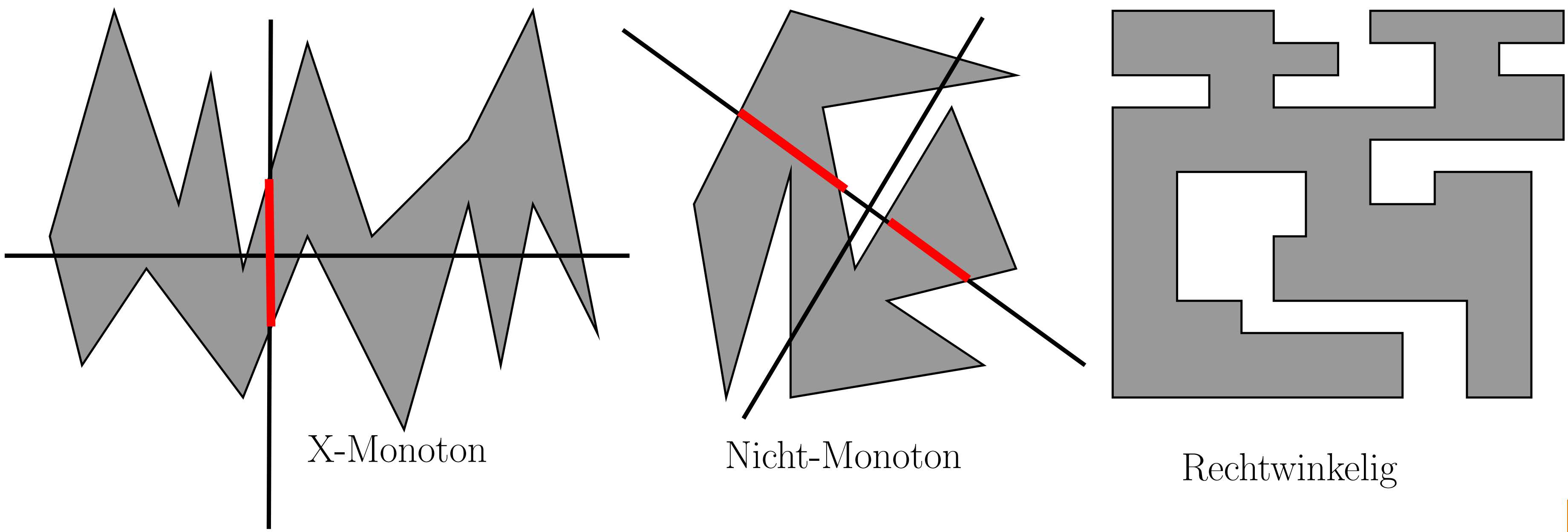
Simple Polygon Offline

- Optimal exploration tour
- Agent with vision, start point s, boundary
- Polygon is fully known
- Depth restriction
- First: General approach. Then: Depth restriction!



Simple polygon SWR Offline

- Standard approach (a bit simpler)
- Monotone polygons: Monoton w.r.t l
- Rectilinear polygons
- Rectilinear and monotone polygons: SWR in $O(n)$



In general: Important segments? Def.

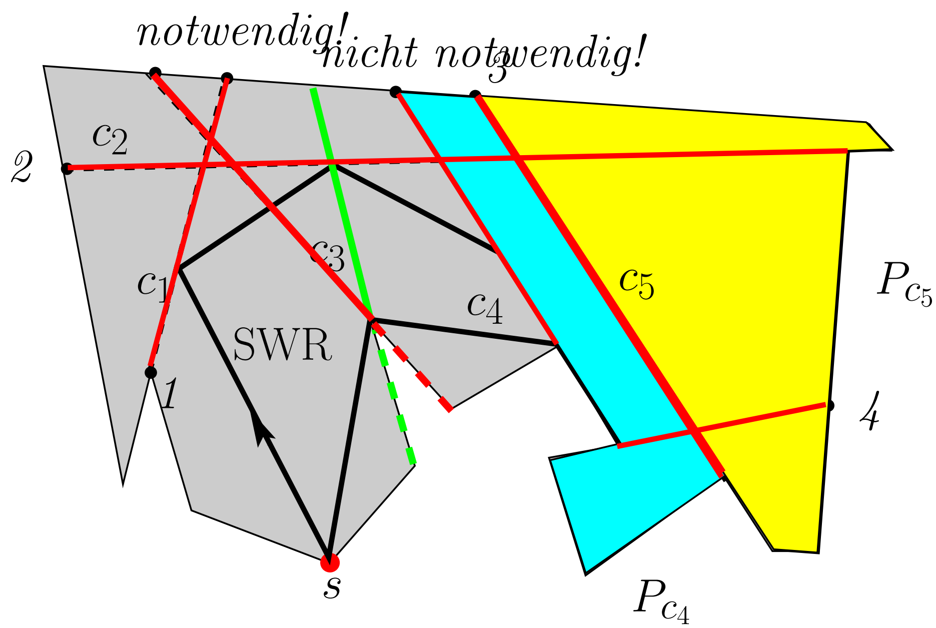
a) (Cuts) Extension of reflex vertex

b) Necessary cuts (w.r.t. s)

c) Dominance-Relationsship $P_{c_i} \subseteq P_{c_j}$

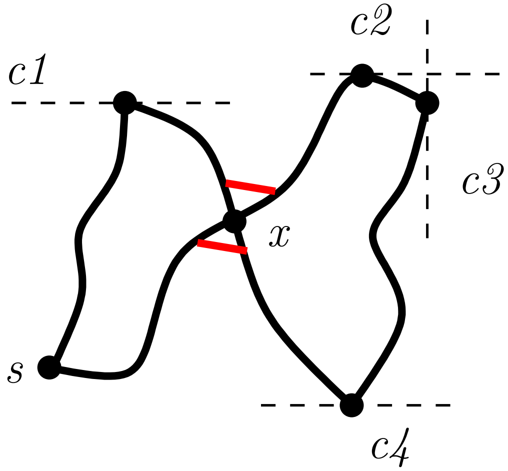
d) Essential cuts

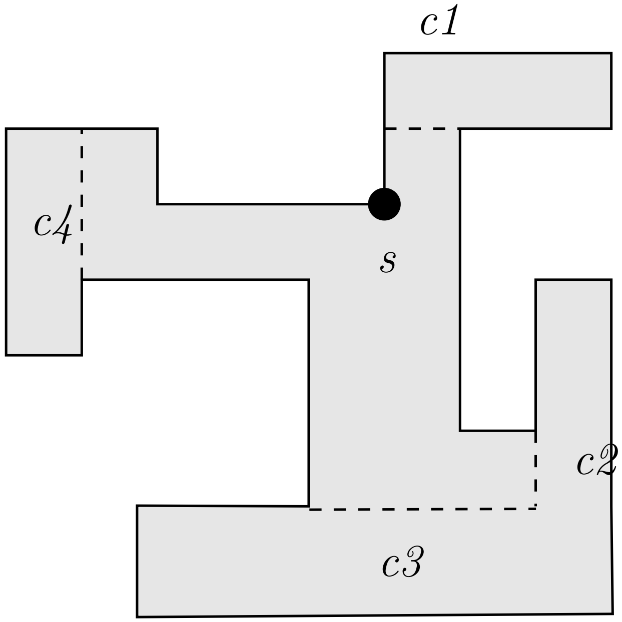
e) Order of the essential cuts



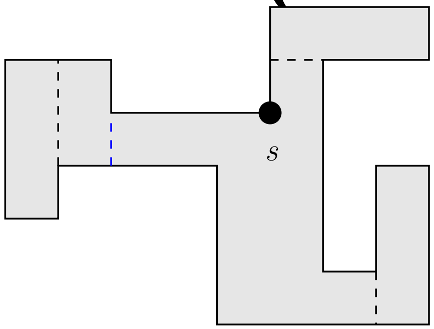
Ordnung alomng the boundary Lem.

- Rectilinear polygon
- Essential cuts intersect at most once
- SWR visits cuts by order around boundary
- Contradiction! Shortcut!
- $O(n)$ Algorithm!!



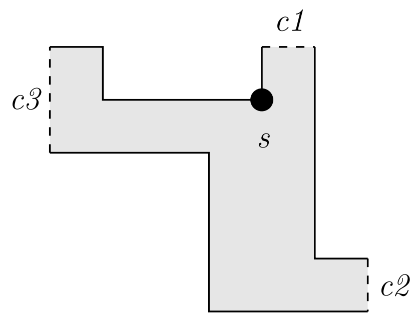


SWR (RW Polygon) $O(n)$ Theo.



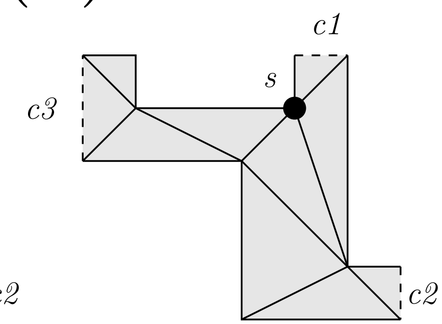
(i) Wesentliche Cuts

O(n)



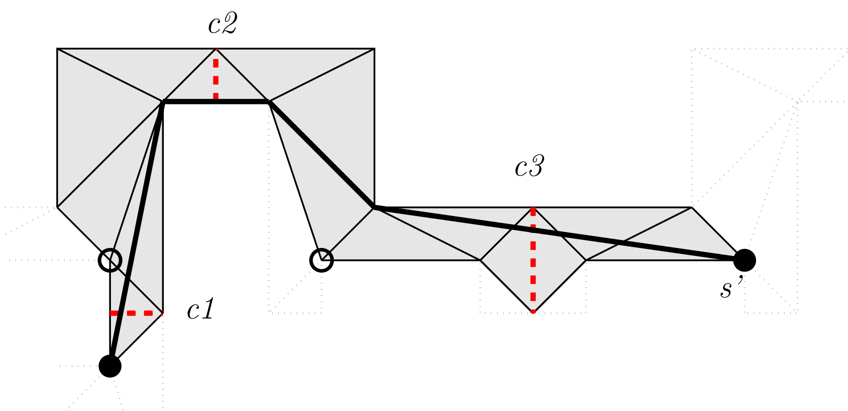
(ii) Abschneiden!

O(n)



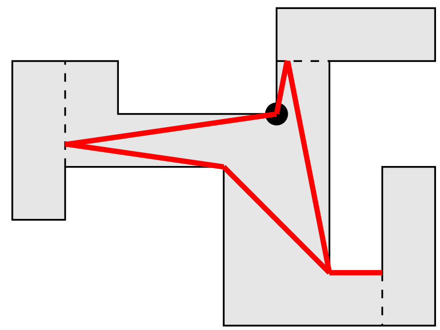
(iii) Triangulation

O(n)



(iv) Spiegeln und Ausrollen!!
(v) Weg berechnen

O(n)

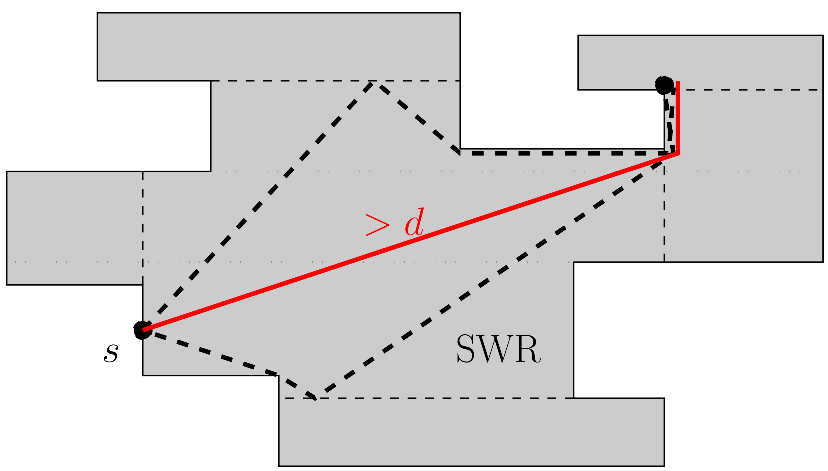


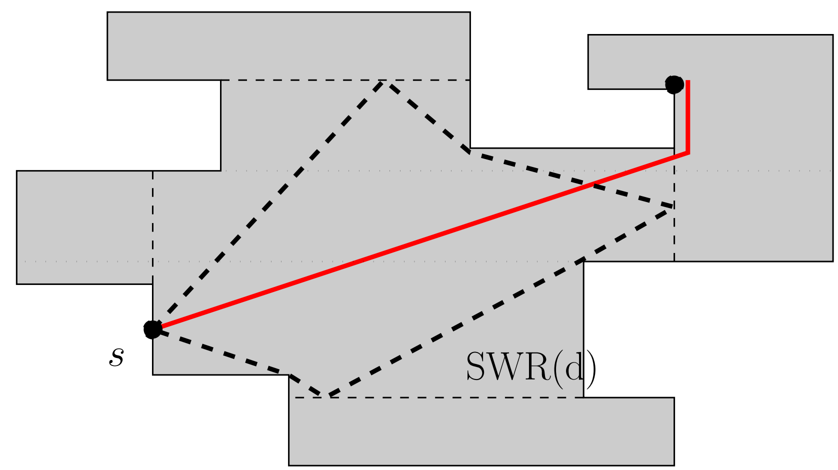
(vi) Zurckklappen!

O(n)

SWR (Rect. polygon) depth restriction?

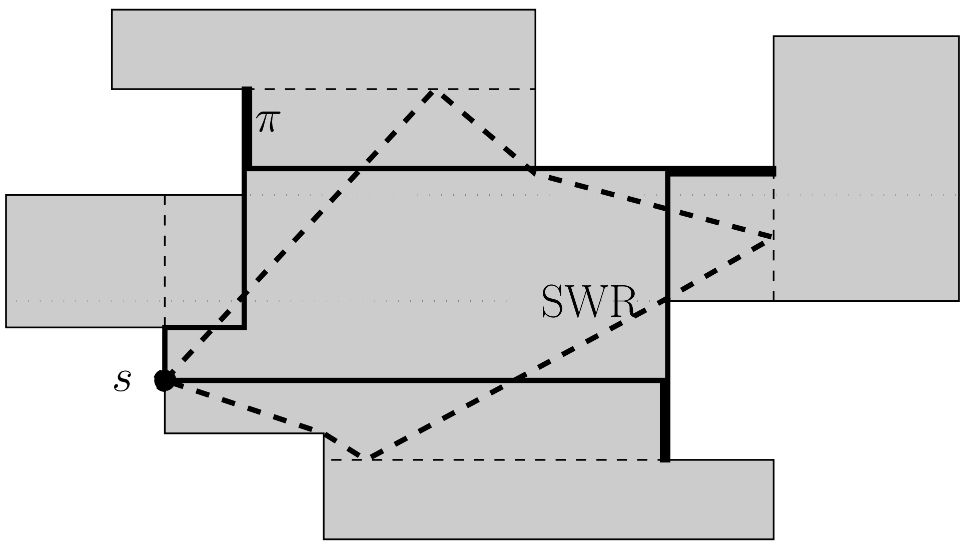
- Ignore cuts with distance $> d$, Shortest path to cut
- Ignore a cut here, Algorithm as before
- $\text{Expl}\,(d) = \text{Expl}_{\text{OPT}}(d)$
- **Theorem**: 8 Approximation of optimal search path!





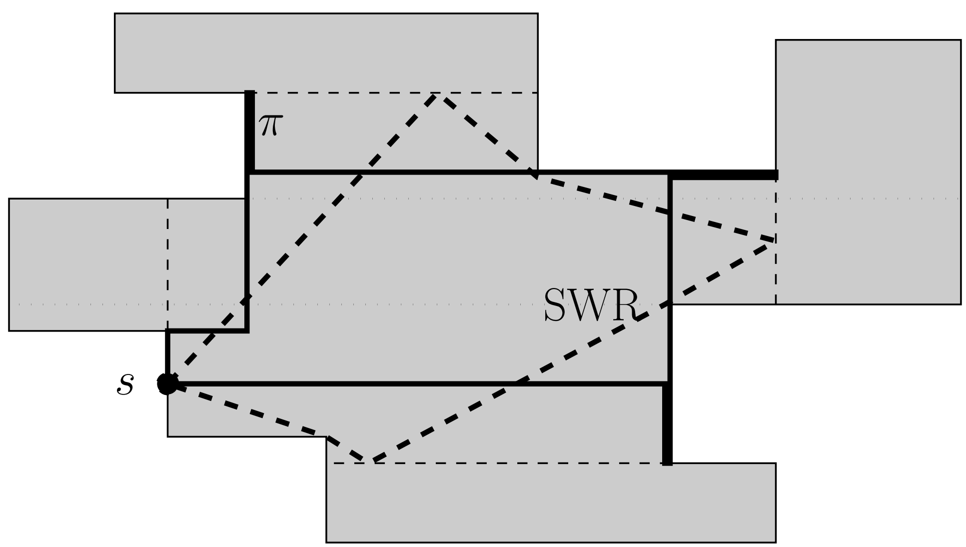
Rectilinear polygons Online

- Agent with vision, start point s
- Szene is not known!
- Depth restriction?
- First: General approach. Then: Depth restriction!



Rectilinear polygons Online

- Assume, s boundary point
- Greedy! Scan the boundary up to the first invisible point. Move to the cut on the shortest path!
- Shortest L_1-path to the cut, online!
- **Algorithmus** Always approach the next reflex vertex along the boundary that blocks the visibility.



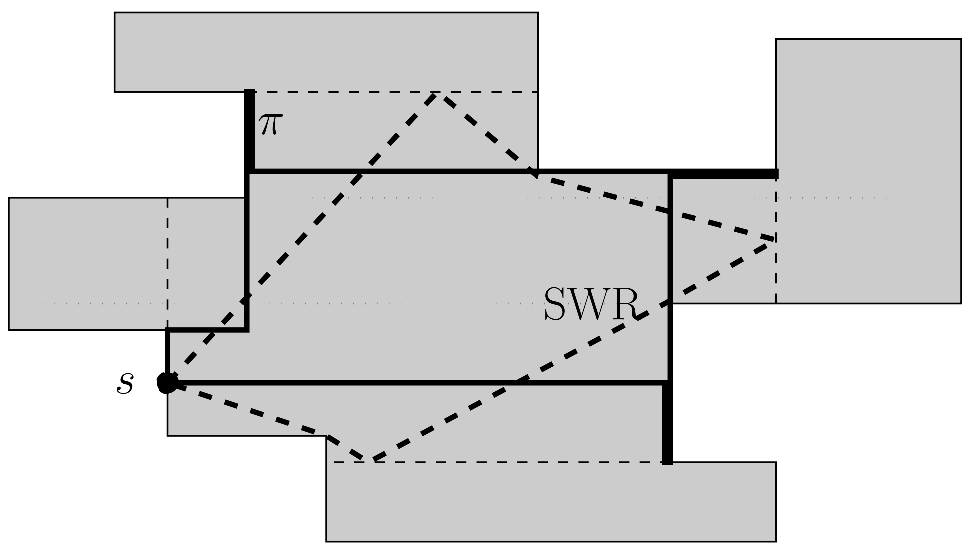
Online variant for rectilinear polygons

Exploration rectilinear polygone DKP

WHILE Polygon not fully explored

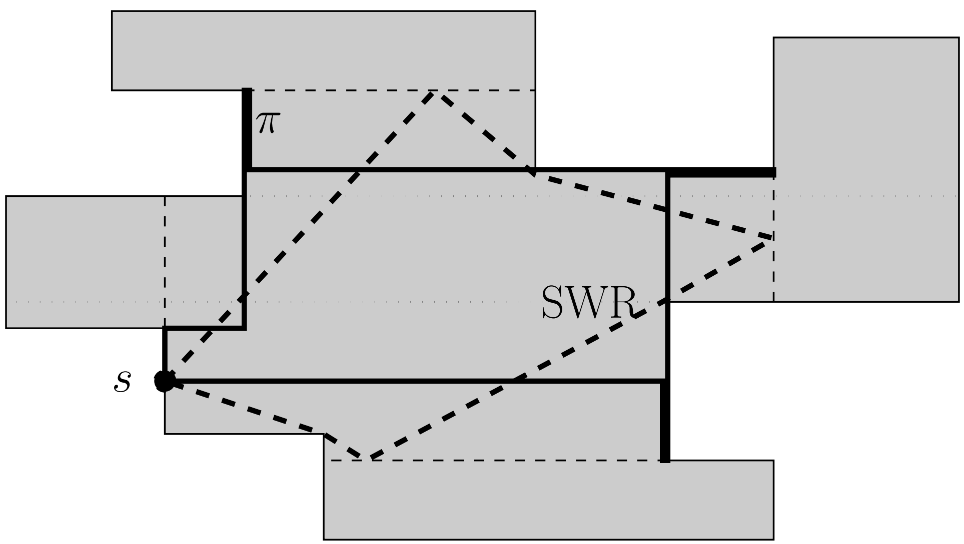
Do Move orthogonally toward the cut of next reflex vertex in cw-order along the boundary

END



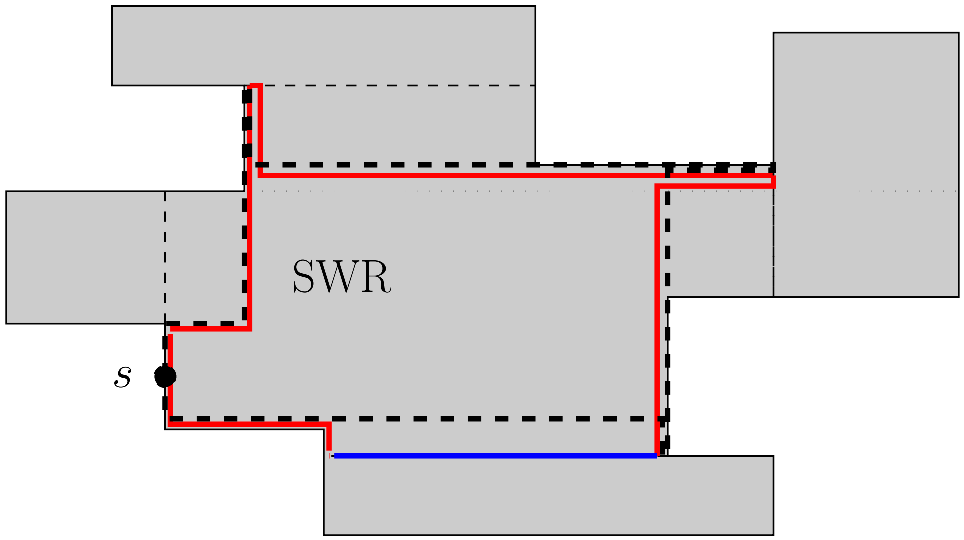
L_1-opt./$\sqrt{2}$-competitive! Theorem

- Analysis: 1) Show L_1-optimal path to essential cuts
- Induktively! Number of steps! Fisrt step, trivial!
- Ass.: Along opt. L_1-path to an essential cut!
- Next step, visit cut, ok! Otherwise, vertex on the track! Next step also optimal!



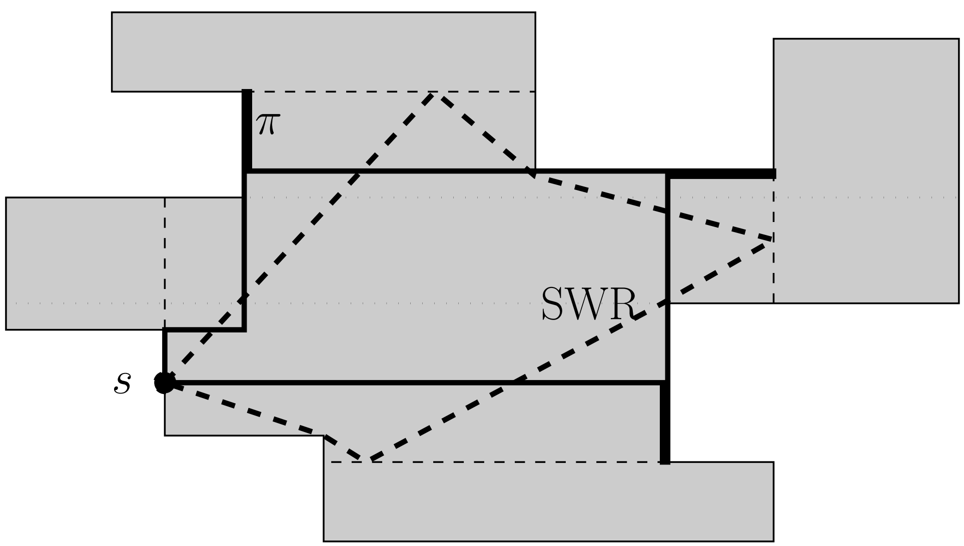
L_1-opt./$\sqrt{2}$-competitive! Theorem

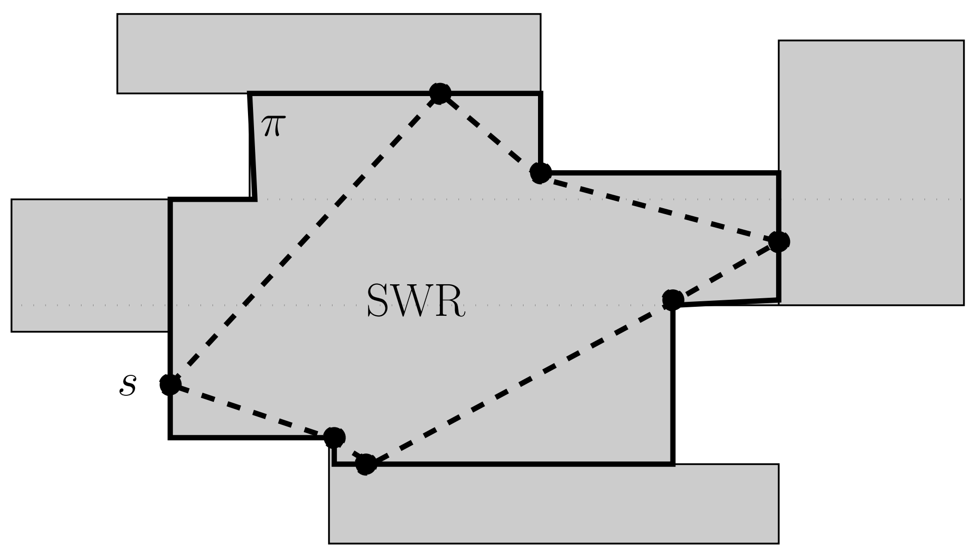
- Sketch!
- Analysis: 2) Combine the optimal L_1-paths!
- L_1-paths, combination is also L_1-optimal!



L_1-opt./$\sqrt{2}$-competitive! Theorem

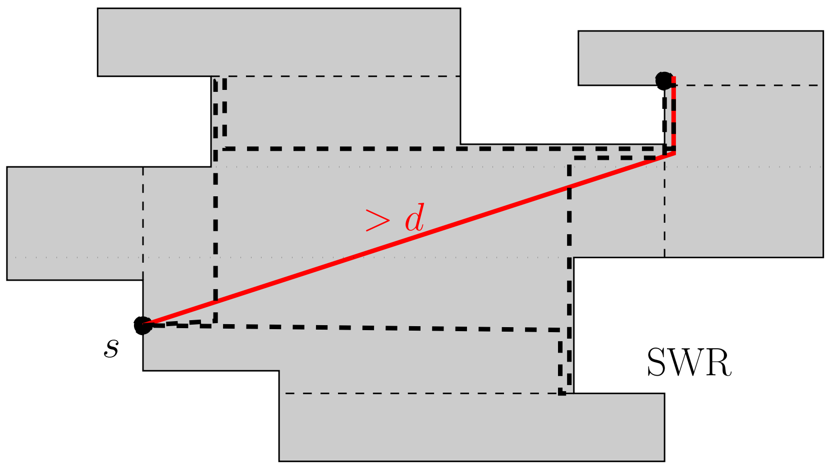
- *Shift* paths toward the cuts, such that (Euclidean) SWR is included! Path has the same length!
- L_1-optimal path between any two points!
- Euclidean shortest path in between
- Triangle! Situation! Blackboard! $\sqrt{2}$-Umweg maximal!

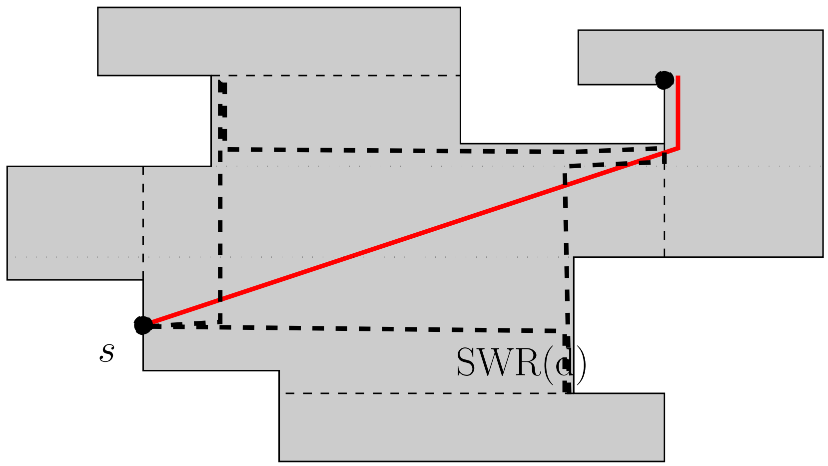




L_1-opt./$\sqrt{2}$-competitive! Theorem

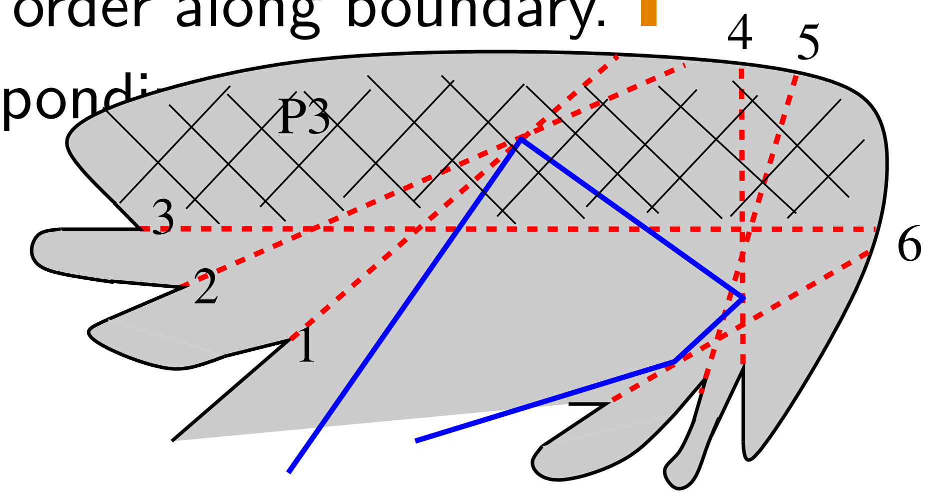
- $\sqrt{2}$-competitive
- Depth restrictable!
- Online: Ignore Cuts with distance $> d$
- $\text{Expl}_{\text{ONL}}(d) \leq \sqrt{2}\ \text{Expl}_{\text{OPT}}(d)$
- **Theorem**: $8\sqrt{2}$-Approximation





SWR (General case): Offline!

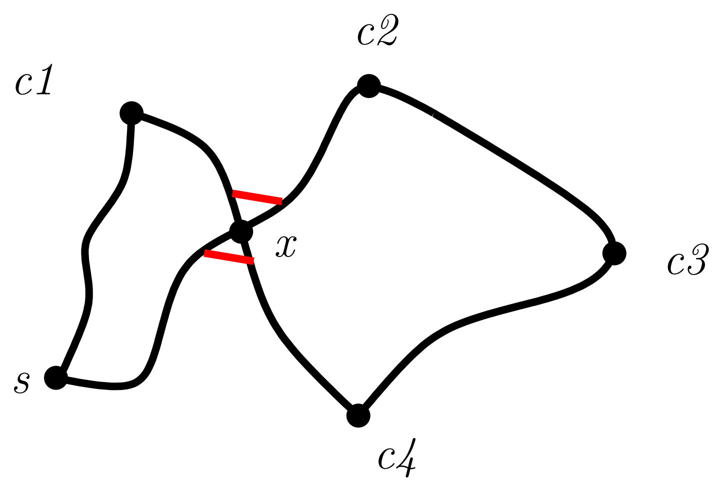
- Corner problem!!
- Sequence of essential cuts, successive cuts
- Not visited by order along boundary.
- But the correspondi



Visiting the corners!

The SWR visits the different corners by the order along the boundary.

Proof: As before! Local shortcuts!



Adjustments inside the corners: Not easy to realize!