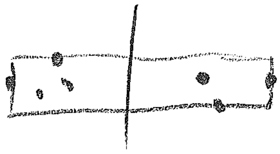
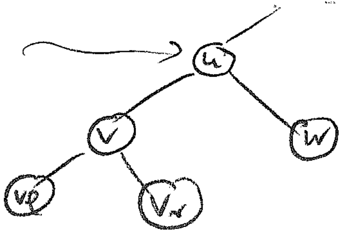


Recap WSPD

* build recursively split tree $T(S)$
  $O(n)$ size, $O(n \log n)$ time

* for each



in $T(S)$ call FindPairs (v,w)

if S_v, S_w w.s. w.r.t. s: report, return

else if S_v has longer bounding box than S_w:
FindPairs (v_ℓ, w), FindPairs (v_r, w)

$\rightsquigarrow$ WSPD consisting of m w.s. pairs A_i, B_i s.th.

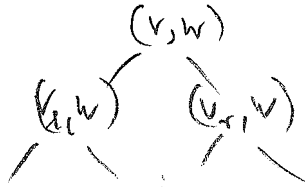
$$S^2 \subseteq \bigcup_{i=1}^{m} A_i \times B_i \cup \bigcup_{i=1}^{m} B_i \times A_i$$

running time ?

* \# calls to FindPairs $\in O(m)$:

Proof: for each node u in $T(S)$
$v \quad w$

recursion tree



all leaves of these trees $\Longleftrightarrow$ well-separated pairs found (each only once)

* $m \in O(n)$ if dimension d and sep. constant s are fixed

main proof idea: show there are only $O(1)$ sets B_i s.th. pair (A, B_i) is reported, for each A (only $O(n)$ many)

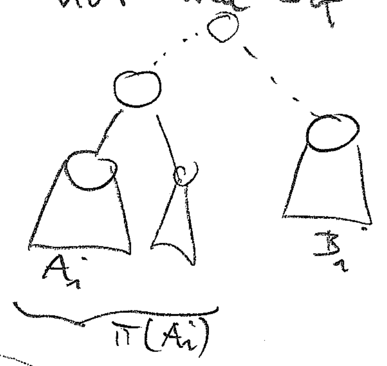
clear: $(A, B_i), (A, B_j)$ reported $\Rightarrow$ B_i, B_j separated by hyperplane, bounding boxes disjoint

($\forall$ nodes v, w in $T(S)$: $S_v \subseteq S_w$ or $S_w \subseteq S_v$ or sep.)

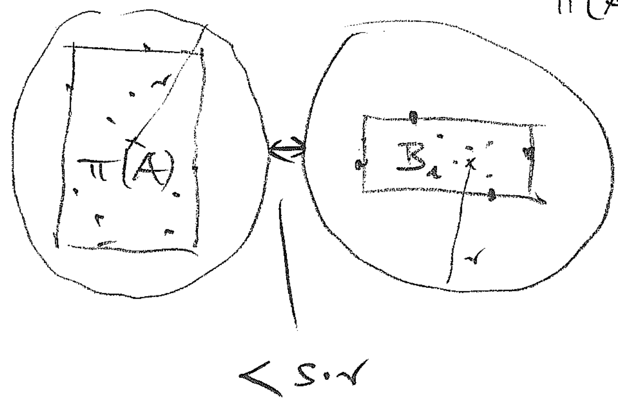
moreover: pair (A, B_i) reported as well-separated

$\Rightarrow$ pair $(\pi(A), B_i)$ not well sep (or vice versa)

father



$\Rightarrow$



$\Rightarrow$ the boxes of all B_i are close to each other

now use packing argument.

<u>Theorem</u> WSPD of n points in $\mathbb{R}^d$: $O(n \log n)$ time

of size $m \in O(n)$ $O(n)$ space

<u>Applications</u>

Closest pair in time $O(m) \subset O(n)$ ins ✓

for each $p \in S$, all k nearest neighbors : $O(n \log n + nk)$

($V(S)$ can do this only for $k = 1$) (without proof)

post office? given arbitrary $q \in \mathbb{R}^d$, report nearest $p \in S$

curse of dimensionality

only approximate solutions known

(report p' s.th. $|qp'| \leq (1+\varepsilon)\, |qp|$

using dynamic structures)

main WSPD application: construction of good geometric networks connecting points in S

good: low dilation $\delta(N) := \min_{p,q \in S} \frac{|\pi_N(pq)|}{|pq|}$

few edges

	dilation	# edges
complete graph:	1, optimal	$\binom{n}{2} \in \Theta(n^2)$
(min) spanning tree:	can be in $\Omega(n)$	$n-1$, optimal

?

Small miracle: Construct WSPD for S wrt $s>4$: $O(n \log n)$

for each w.s. pair A_i, B_i:
Pick $p_i \in A_i$, $q_i \in B_i$
add edge $\overline{p_i q_i}$

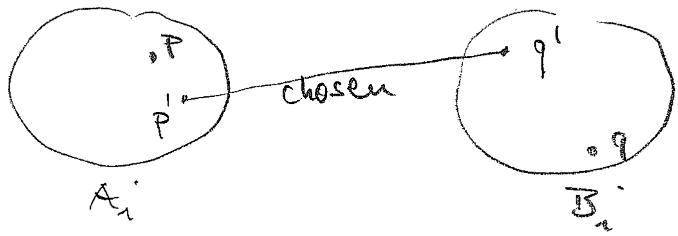
$\leadsto$ $O(m) \subseteq O(n)$ edges (by previous Theorem)

dilation $\leq \frac{s+4}{s-4}$ $\searrow 1$ as $s \to \infty$ (by induction on rank of $|pq|$)

induct. basis

pq closest pair $\Rightarrow \exists i: A_i = \{p\}$ $B_i = \{q\}$

induct. step



$|pp'|$, $|qq'|$ $< \frac{2}{s}|pq| \underset{\text{Lemma 1}}{\Longrightarrow} \exists$ good paths by induction $\}$ concatenation does it.

$|p'q'| < (1+\frac{4}{s})|pq|$