Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Expected Search Number

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Expected number of vertices saved, Definitions

- $G = (V, E)$ fixed number k of agents
- k-surviving rate, $s_k(G)$:
  Expectation of the *proportion* of vertices saved
- Any vertex root vertex with probability $\frac{1}{|V|}$
- Classes, C, of graphs G:
  For constant ϵ, $s_k(G) \geq \epsilon$
- Given G, k, $v \in V$:
  $\mathrm{sn}_k(G, v)$: Number of vertices that can be
  protected by k agents, if the fire starts at v
- Goal: $\frac{1}{|V|} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \epsilon |V|$
- Class C: Minimum number k that
  guarantees $s_k(G) > \epsilon$ for any $G \in C$
  The firefighter-number, $\mathrm{ffn}(C)$, of C.

Expected number of vertices saved

Firefighter-Number for a class C of graphs:
Instance: A class C of graphs $G = (V, E)$.
Question: Assume that the fire breaks out at any vertex of a graph $G \in C$ with the same probability. Compute $\text{ffn}(C)$.

$\text{ffn}(C)$ for trees? For stars?

Planar graph: $\text{ffn}(C) \geq 2$, bipartite graph $K_{2,n-2}$.

Main Theorem: For planar graphs we have $2 \leq \text{ffn}(C) \leq 4$

Idea for the upper bound ffn$(C) \leq 4$

- Vertices subdivided into classes X and Y
- $r \in X$ allows to save many (a linear number of) vertices
- $r \in Y$ allows to save only few (almost zero) vertices
- Finally, $|Y| \leq c|X|$ gives the bound
- Simpler result first!

Simple proof!

Theorem 43: For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- Euler formula, $c + 1 = v - e + f$, for planar graphs, e edges, v vertices, f faces and c components
- Planar graph with no 3- and 4-cycle has average degree less than $\frac{10}{3}$
- Assume $\frac{10}{3}v \leq 2e$! Which is $v \leq \frac{3}{5}e$
- Also conclude $5f \leq 2e$.
- Insert, contradiction!
- Similar arguments: A graph with no 3-, 4 and 5-cylces has average degree less than 3!

Subdivision into X and Y

Theorem 46: For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

Subdivide the vertices V of G into groups w.r.t. the degree and the neighborship

- Let X_2 denote the vertices of degree ≤ 2.
- Let Y_4 denote the vertices of degree ≥ 4.
- Let X_3 denote the vertices of degree exactly 3 but with at least one neighbor of degree ≤ 3.
- Let Y_3 denote the vertices of degree exacly 3 but with all neighbors having degree > 3 (degree 3 vertices not in X_3).

Let x_2,x_3,y_3 and y_4 denote cardinality of the sets

Counting the portion for X

Theorem 46: For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- $|V| = n$, $x_2 + x_3 + y_3 + y_4 = n$
- $v \in X_2$: save $n-2$ vertices
- $v \in X_3$: save $n-2$ vertices
- For starting vertices in Y_3 and Y_4, we assume that we can save nothing!
- Show: $s_2(G) \cdot n = \frac{1}{n} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \epsilon \cdot n$

$$\frac{1}{n^2} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \frac{1}{n^2}(x_2 + x_3)(n-2) = \frac{n-2}{n} \cdot \frac{x_2 + x_3}{x_2 + x_3 + y_3 + y_4}$$

Relationsship between X and Y

Theorem 46: For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- Fixed relation between $x_2 + x_3$ and $y_3 + y_4$
- First: Correspondance between Y_3 and Y_4
- $G_Y = (V_Y, E_Y)$: Edges of G with one vertex in Y_3 and one vertex in Y_4 (degree at least 4)
- $3y_3$ edges, at most $y_3 + y_4$ vertices, bipartite
- Cylce: Forth and back from Y_3 to Y_4
- No cycle of size 5!
- Average degree of vertices of G_Y is at most 3
- Counting $3(y_3 + y_4)$, counts at least any edge twice, so $3(y_3 + y_4) \geq 6y_3$
- $y_3 \leq y_4$

Counting edges by vertex degrees

Theorem 46: For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- Fixed relation between $x_2 + x_3$ and $y_3 + y_4$, $y_3 \leq y_4$
- Counting $\frac{10}{3}(x_2 + x_3 + y_3 + y_4)$ edges we have at least counted $3x_3 + 3y_3 + 4y_4$ edges
- $9x_3 + 9y_3 + 12y_4 \leq 10(x_2 + x_3 + y_3 + y_4)$
- $2y_4 - y_3 \leq 10x_2 + x_3$
- By $y_3 \leq y_4$ we have $y_4 \leq 10x_2 + x_3$
- Finally: $y_3 + y_4 \leq 20x_2 + 2x_3 \leq 20(x_2 + x_3)$

Use the inequality!

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Finally: $y_3 + y_4 \leq 20x_2 + 2x_3 \leq 20(x_2 + x_3)$

$$\frac{n-2}{n} \cdot \frac{x_2 + x_3}{x_2 + x_3 + y_3 + y_4} \geq \frac{n-2}{n} \cdot \frac{x_2 + x_3}{21(x_2 + x_3)} = \frac{n-2}{21n}. \qquad (1)$$

- $n = 2$: one vertex distinct from the root
- $3 \leq n \leq 44$: at least $\frac{2}{44}$
- $n \geq 44$: $s_2(G) \geq \frac{42}{21 \cdot 44} = \frac{1}{22}$.
- Expected value of saved vertices is always $\frac{1}{22}n$.

Theorem 47: Using four firefighters in the first step and then always three firefighters in each step, for every planar graph G there is a strategy such that $s_4(G) \geq \frac{1}{2712}$ holds.

- Maximal, planar without multi-edges.
- Triangulation, any face has exactly 3 edges
- Subdivide V of G into sets X and Y.
- X set of vertices strategy that save at least $n-6$ vertices
- For Y we do not expect to save any vertex, for $|V| = n$
- Final conclusion: For $\alpha = \frac{1}{872}$

$$|Y| \leq \left(93 + \frac{3}{\alpha}\right)|X| = 2709|X|. \qquad (2)$$

Warm up for planar graphs

Theorem 47: Using four firefighters in the first step and then always three firefighters in each step, for every planar graph G there is a strategy such that $s_4(G) \geq \frac{1}{2712}$ holds.

$$|Y| \leq \left(93 + \frac{3}{\alpha}\right)|X| = 2709|X|\,. \qquad (3)$$

Thus from $|X| + |Y| = |V| = n$ we conclude

$$s_4(G) \geq \frac{n-6}{n} \cdot \frac{|X|}{|X| + |Y|} \geq \frac{n-6}{n} \cdot \frac{|X|}{2710|X|} = \frac{n-6}{2710n}\,.$$

For $n \geq 10846$ we have

$$s_4(G) \geq \frac{1}{2710} - \frac{6}{4 \cdot 2710^2} \geq \frac{2710 - 3/2}{2710^2} \geq \frac{1}{2712}$$

For $2 \leq n < 10846$ we save at least $\min(4, n-1)$ in the first step, which gives also $s_4(G) \geq \frac{1}{2712}$.

Subdivision into X and Y!

- For degree $3 \leq d \leq 6$ let X_d denote the vertices that guarantee to save at least $|V| - 6$ vertices.
- All other vertices form the set Y_d for $d \geq 5$.

Vertex v of degree $1, 2, 3, 4$ belongs to X!

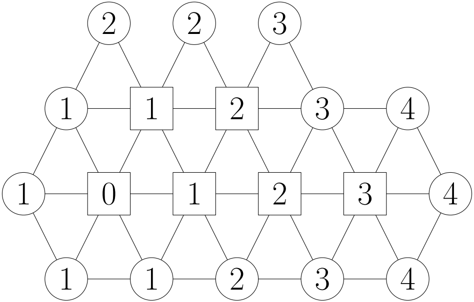
Vertex v of degree 5 with neighbor u of degree at most 6:

$v \in X_5$ by construction, fire spreads to u and is stopped then!

Vertices from Y_6

Lemma 48: For a vertex $v \in Y_6$ there is a path of length at most 3 from v to a vertex u that has degree distinct from v (i.e., $\neq 6$) and the inner vertices of the path have degree exactly 6.

- If not, vertex v belongs to X_6! Build a Hexagon!



Vertices with degree at least 7

Lemma 49: A vertex with $d(v) \geq 7$ has at most $\lfloor \frac{1}{2} d(v) \rfloor$ neighbors in Y_5.

- neighbor $u \in Y_5$ has two neighbors n_1 and n_2 in common with v
- n_1 or n_2, degree at most 6, then $u \in X_5$
- Vertices u from Y_5 around v, *separated* by vertices of degree ≥ 7

Potential for the vertices

Lemma 50: For a simple, maximal planar graph we have

$$\sum_{v \in V} (d(v) - 6) = -12\,. \qquad (4)$$

- maximal, simple planar graph gives $3f = 2e$ (all faces are triangles)
- $\sum_{v \in V} d(v) = 2e$
- Euler formula: $v - e + f = 2$
- $v - e + \frac{2}{3}e = 2 \iff 2e - 6v = -12$

Potential dsitribution!

- *Intitial potential* $p_1(v) := (d(v) - 6)$ of every vertex
- Distribute (cost neutral) to $p_2(v)$
- $\sum_{v \in V} p_1(v) = \sum_{v \in V} p_2(v) = -12$

The rules for the distribution are as follows:

Rule A: A vertex v of degree at least 7 gives a value of $\frac{1}{4}$ to each neighbor vertex from Y_5.

Rule B: For a vertex $v \in Y_6$ we choose exactly one vertex u with $d(u) \neq 6$ and distance $d(v, u) \leq 6$ as in Lemma 48. The vertex u gives a value of $\alpha > 0$ to v.

Lemma 48: For a vertex $v \in Y_6$ there is a path of length at most 3 from v to a vertex u that has degree distinct from v (i.e., $\neq 6$) and the inner vertices of the path have degree exactly 6.

Potential distribution!

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Lemma 50: There is a constant $\alpha > 0$ such that a distribution by Rule A and B gives $\sum_{v \in V} p_1(v) = \sum_{v \in V} p_2(v) = -12$ and for every $v \in X$ we have $p_2(v) > -3 - 93\alpha$ and for every $v \in Y$ we have $p_2(v) \geq \alpha$.

Conclusion: $\alpha = \frac{1}{872}$ will do the job.

$$-12 = \sum_{v \in V} p_2(v) \geq (-3 - 93\alpha)|X| + \alpha|Y|$$

$$|Y| \leq \left(93 + \frac{3}{\alpha}\right)|X| < 2790|X|$$

Theorem 47: Using four firefighters in the first step and then always three firefighters in each step, for every planar graph G there is a strategy such that $s_4(G) \geq \frac{1}{2712}$ holds.

- Maximal, planar without multi-edges.
- Triangulation, any face has exactly 3 edges
- Subdivide V of G into sets X and Y.
- X set of vertices strategy that save at least $n - 6$ vertices
- For Y we do not expect to save any vertex, for $|V| = n$
- Final conclusion: For $\alpha = \frac{1}{872}$

$$|Y| \leq \left(93 + \frac{3}{\alpha}\right)|X| = 2709|X|\,. \tag{5}$$

Theorem 47: Using four firefighters in the first step and then always three firefighters in each step, for every planar graph G there is a strategy such that $s_4(G) \geq \frac{1}{2712}$ holds.

$$|Y| \leq \left(93 + \frac{3}{\alpha}\right)|X| = 2709|X|\,. \tag{6}$$

Thus from $|X| + |Y| = |V| = n$ we conclude

$$s_4(G) \geq \frac{n-6}{n} \cdot \frac{|X|}{|X| + |Y|} > \frac{n-2}{n} \cdot \frac{|X|}{2710|X|} = \frac{n-6}{2710n}\,.$$

For $n \geq 10846$ we have

$$s_4(G) \geq \frac{1}{2710} - \frac{6}{4 \cdot 2710^2} \geq \frac{2710 - 3/2}{2710^2} \geq \frac{1}{2712}$$

For $2 \leq n < 10846$ we save at least $\min(4, n-1)$ in the first step, which gives also $s_4(G) \geq \frac{1}{2712}$.

Rule B: Potential distribution!

Rule B: For a vertex $v \in Y_6$ we choose exactly one vertex u with $d(u) \neq 6$ and distance $d(v,u) \leq 6$ as in Lemma 48. The vertex u gives a value of $\alpha > 0$ to v.

How often can a vertex u with $d(u) \neq 6$ give a potential of α to some vertex v? Rough upper bound with respect to the maximal distance ≤ 3 from u.

- Distance 1: $d(v)$ times to a direct neighbor, if all of them are in Y_6. This gives $1 \cdot d(u)$.
- Distance 2: For all $d(v)$ neighbors of the first case, at most 5 times, if the $d(v)$ neighbors of the above case have degree 6 and all 5 remaining neigbors are from Y_6. This gives $5 \cdot d(u)$.
- Distance 3: For all vertices of the second case, at most 5 times, if the vertices of the second case all have degree 6 and the remaining neighbors are from Y_6. This gives $25 \cdot d(u)$.

Rule B: Potential distribution!

Altogether, any vertex u with $d(u) \neq 6$ can give a potential α at most $(1+5+25)d(u) = 31d(u)$ times.

Upper bounds for the potential $p_2(v)$:

- $v \in X_3$: We have $p_2(v) \geq -3 - 93\alpha$ because $d(v) = 3$ and $p_1(v) = -3$.
- $v \in X_4$: We have $p_2(v) \geq -2 - 124\alpha$ because $d(v) = 4$ and $p_1(v) = -2$.
- $v \in X_5$: We have $p_2(v) \geq -1 - 155\alpha$ because $d(v) = 5$ and $p_1(v) = -1$.

Vertices of degree 6:

- $v \in X_6$: $p_2(v) = 0$ because $d(v) = 6$ and $p_1(v) = 0$.
- $v \in Y_6$: $p_2(v) = p_1(v) + \alpha = \alpha$
  Rule B gives a single value α from some u to v, and by Lemma 48 such a vertex has to exist.

Rule B: Potential distribution!

Vertices of degree 6:

- $v \in X_6$: $p_2(v) = 0$ because $d(v) = 6$ and $p_1(v) = 0$.
- $v \in Y_6$: $p_2(v) = p_1(v) + \alpha = \alpha$
  Rule B gives a single value α from some u to v, and by Lemma 48 such a vertex has to exist.

Rule A: Potential distribution!

Rule A: A vertex v of degree at least 7 gives a value of $\frac{1}{4}$ to each neighbor vertex from Y_5.
(No more than $\left\lfloor \frac{1}{2} d(v) \right\rfloor$ by Lemma 49!)

Vertex v and $d(v) \geq 7$

$$p_2(v) \geq (d(v) - 6) - \left\lfloor \frac{1}{2} d(v) \right\rfloor \cdot \frac{1}{4} - 31 d(v) \alpha \, .$$

So the remaining cases can be estimated by

- $v \in X \cup Y$ with $d(v) = 7$: $p_2(v) \geq \frac{1}{4} - 217\alpha$.
- $v \in X \cup Y$ with $d(v) \geq 8$: $p_2(v) \geq d(v) \left(\frac{7}{8} - 31\alpha \right) - 6$ by $\lfloor \frac{1}{2} d(v) \rfloor \cdot \frac{1}{4} \leq \frac{1}{8} d(v)$.

$\alpha = \frac{1}{218 \cdot 4} = \frac{1}{872}$ gives $p_2(v) \geq \alpha$

Remaining vertices!

$\alpha = \frac{1}{218 \cdot 4} = \frac{1}{872}$ gives $p_2(v) \geq -\alpha - 93\alpha$

Upper bounds for the potential $p_2(v)$:

- $v \in X_3$: We have $p_2(v) \geq -3 - 93\alpha$ because $d(v) = 3$ and $p_1(v) = -3$.
- $v \in X_4$: We have $p_2(v) \geq -2 - 124\alpha$ because $d(v) = 4$ and $p_1(v) = -2$.
- $v \in X_5$: We have $p_2(v) \geq -1 - 155\alpha$ because $d(v) = 5$ and $p_1(v) = -1$.

Vertices of degree 6:

- $v \in X_6$: $p_2(v) = 0$ because $d(v) = 6$ and $p_1(v) = 0$.
- $v \in Y_6$: $p_2(v) = p_1(v) + \alpha = \alpha$
  Rule B gives a single value α from some u to v, and by Lemma 48 such a vertex has to exist.

Proof of Lemma 50

Lemma 50: There is a constant $\alpha > 0$ such that a distribution by Rule A and B gives $\sum_{v \in V} p_1(v) = \sum_{v \in V} p_2(v) = -12$ and for every $v \in X$ we have $p_2(v) > -3 - 93\alpha$ and for every $v \in Y$ we have $p_2(v) \geq \alpha$.

Overall conclusion:
Theorem 47: Using four firefighters in the first step and then always three firefighters in each step, for every planar graph G there is a strategy such that $s_4(G) \geq \frac{1}{2712}$ holds.

Lemma 50:

Connnected Search vs. non-connected search

- Non-connected, other rules!
- Differ in a factor of 2

1. Place a team of p guards on a vertex.
2. Move a team of m guards along an edge.
3. Remove a team of r guards from a vertex.

Connnected Search vs. non-connected search

D_k denote a tree with root r of degree three and three full binary trees, B_{k-1}, of depth $k-1$ connected to the r.

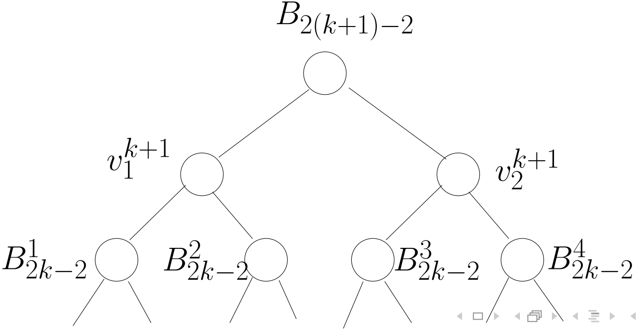
Lemma 31: For the graph D_k, we conclude $\mathtt{cs}(D_k) = k + 1$.

- Consider T_1, T_2 and T_3 at r!

D_k denote a tree with root r of degree three and three full binary trees, B_{k-1}, of depth $k-1$ connected to the r.

Lemma 32: For D_{2k-1} we conclude $s(D_{2k-1}) \leq k+1$.

- $k = 1$ is trivial. So assume $k > 1$
- Place one agent at the root r and successively clean the copies of B_{2k-2} by k agents
- This is shown by induction!



Corollary 33: There exists a tree T so that $cs(T) \leq 2s(T) - 2$ holds.

$T = D_{2k-1}$, $s(D_{2k-1}) \leq k + 1$, $\mathrm{cs}(D_{2k-1}) = 2k$

$$\frac{cs(T)}{s(T)} < 2 \text{ for all trees } T\,.$$