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Mathematisch-Naturwissenschaftliche Fakultät

Theoretical Aspects of Intruder Search

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

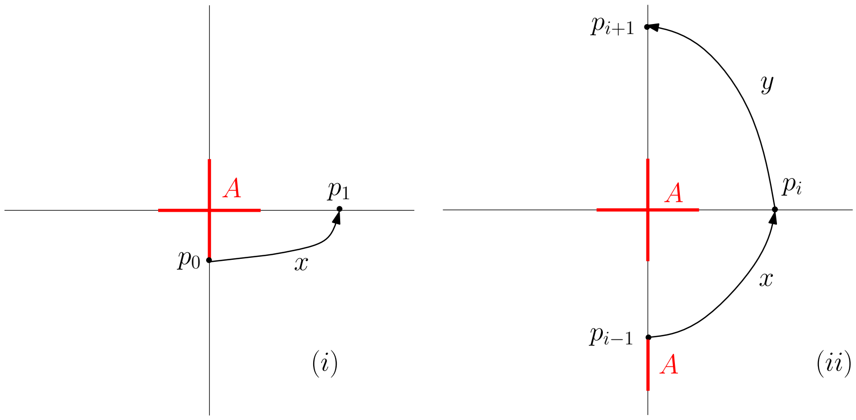

Figure 5.14: Proof of Theorem 57.

The running time of the overall last spiral is smaller than $\frac{1}{\cos\beta_m}e^{2\pi\cot\beta_m}$. Additionally, moving to point $(1,0)$ yields $e^{2\pi\cot\beta_m}-1$. For reaching point $(1,0)$ the fire also has to a distance arbitrarily close to $\frac{1}{\cos\beta_m}$ for following the last spiral, this follows from scaling. If β_m grows, there will be some β_m that finally fulfills

$$\frac{1}{\cos\beta_m} > \frac{1}{\cos\beta_1}\left(\frac{1}{\cos\beta_m}e^{2\pi\cot\beta_m} + (e^{2\pi\cot\beta_m}-1)\right) \tag{5.14}$$

which gives the conclusion. For example for $\beta_1 \approx 1.191388\ldots$ and $\frac{1}{\cos\beta_1} = 2.7$ we require $\beta_m > 1.4268$.

Theorem 56 *(Bressan et al. 2008) For any speed $v > v_l \approx 2.614430844$ there is a spiralling strategy that finally encloses an expanding circle that expands with unit speed.*

5.5 A simple lower bound for spiralling strategies

The following results stem from Klein et al. 2015. A barrier building strategy S is called *spiralling* if it starts on the boundary of a fire of radius A, and visits the four coordinate half-axes in counterclockwise order and at increasing distances from the origin.

Now let S be any spiralling strategy of maximum speed $v \le (1+\sqrt{5})/2 \approx 1.618$, the golden ratio. We can assume that S proceeds at constant speed v. Let $p_0, p_1, p_2, \ldots$ denote the points on the coordinate axes visited, in this order, by S. The following Lemma shows that S cannot succeed because there is still fire burning outside the barrier on the axis previously visited.

Lemma 57 *Let A be the initial fire radius. When S visits point p_{i+1}, the interval $[p_i, p_i + \texttt{sign}(p_i)A]$ on the axis visited before is on fire.*

Proof. The proof is by induction on i. Suppose strategy S builds a barrier of length x between p_0 and p_1, as shown in Figure 5.14 (i). During this time the fire advances x/v along the positive X-axis, so that $A + x/v \le p_1 \le x$ must hold, or

$$\frac{x}{v} \ge \frac{1}{v-1}A > A;$$

the last inequality follows from $v < 2$. Thus, the fire has enough time to move a distance of A from p_0 downwards along the negative Y-axis.

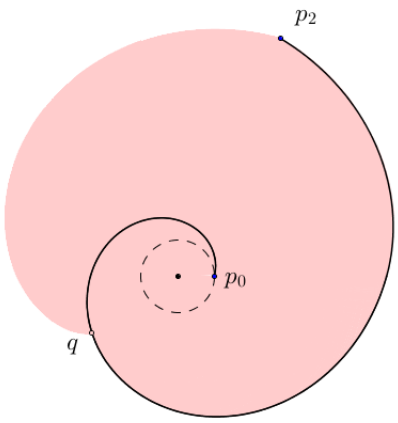

Figure 5.15: The race between the fire and the fighter. When the fighter arrives at point p_2, having constructed a firebreak from p_0 to p_2, the fire has expanded along the outer side of the barrier up to point q. Will the fire fighter be able to contain the fire?

Now let us assume that strategy S builds a barrier of length y between p_i and p_{i+1}, as shown in Figure 5.14 (ii). By induction, the interval of length A below p_{i-1} is on fire. Also, when the fighter moves on from p_i, there must be a burning interval of length at least $A + x/v$ on the positive Y-axis which is not bounded by a barrier from above. This is clear if p_{i+1} is the first point visited on the positive Y-axis, and it follows by induction, otherwise. Thus, we must have $A + x/v + y/v \leq p_{i+1} \leq y$, hence

$$\frac{y}{v} \geq \frac{1}{v-1}A + \frac{1}{v(v-1)}x > A + x.$$

The rightmost inequality follows since v is supposed to be smaller than the golden ratio, which satisfies $X^2 - X - 1 = 0$; hence, $v^2 - v < 1$. This shows that the fire has time to crawl along the barrier from p_{i-1} to p_i, and a distance A to the right, as the fighter moves to p_{i+1}, completing the proof of Theorem 58. □

Theorem 58 *In order to enclose the fire, a spiralling strategy must be of speed*

$$v > \frac{1+\sqrt{5}}{2} \approx 1.618,$$

the golden ratio.

5.6 An alternative approach: The firefighter curve

In this section we describe an alternative approach and a special strategy for the upper bound $v \approx 2.614430844$. The result also stems from Klein et al. 2015. We construct a barrier curve that always keeps as close as possible to the fire. In the beginning this gives a logarithmic spiral of excentricity α for speed $v = \frac{1}{\cos\alpha}$.

In our case while the fighter keeps building the barrier, the fire is coming after her along the outside of the barrier, as shown in Figure 5.15. Intuitively, the fighter can only win this race, and contain the fire, if the last coil of the barrier hits the previous coil. In the example in Figure 5.16, this happens in the second round if $v = 4.1932$; but for smaller values of v, more rounds may be necessary. In this section we prove the following result.

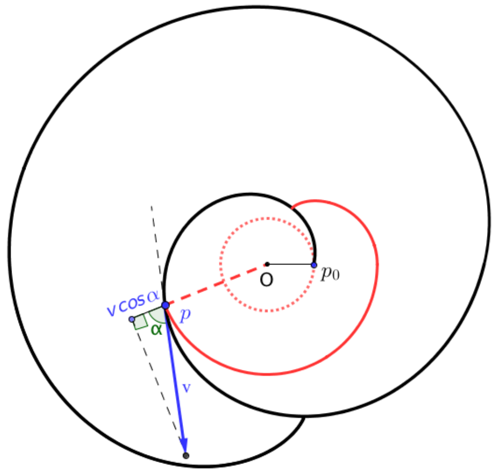

Figure 5.16: At speed $v = 4.1932$ the fire will be fully contained by the fire fighter's barrier in the second round. Angle α is constant because $v \cos\alpha$ equals 1, the fire's expansion speed, by definition of strategy FF.

Theorem 59 *Strategy FF contains the fire if $v > v_c \approx 2.6144$ holds.*

5.6.1 The first rounds

Let p be a point on the barrier curve's first round, as depicted in Figure 5.16, (ii). If α denotes the angle between the fighter's velocity vector at p and the ray from 0 through p, the fighter moves at speed $v \cos\alpha$ away from 0. This implies $v \cos\alpha = 1$, because the fire expands at unit speed and the fighter stays on its frontier, by definition of strategy FF. Since the fighter is operating at constant speed v, angle α is constant, and given by $\alpha = \cos^{-1}(1/v)$.

Consequently, the first part of the barrier curve, between points p_0 and p_1 shown in Figure 5.17, (i), is part of a logarithmic spiral of excentricity α centered at 0. In polar coordinates, this segment can be desribed by $(\varphi, A \cdot e^{\varphi \cot\alpha})$, where $\varphi \in [0, 2\pi]$, and A denotes the distance from the origin to p_0, i.e., the fire's intitial radius.

In general, the curve length of a logarithmic spiral of excentricity α between two points at distance $d_1 < d_2$ to its center is known to be $\frac{1}{\cos\alpha}(d_2 - d_1)$. Thus, we have for the length l_1 of the barrier curve from p_0 to p_1 the equation

$$l_1 \;=\; \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1) \tag{5.15}$$

From point p_1 on, the geodesic shortest path, along which the fire spreads from 0 to the fighter's current position, p, is no longer straight. It starts with segment $0p_0$, followed by segment p_0p, until, for $p = p_2$, segment p_0p becomes tangent to the barrier curve at p_0; see Figure 5.17, (ii). By the same arguments as above, between p_1 and p_2 the barrier curve constructed by FF is also part of a logarithmic spiral of excentricity α, but now centered at p_0. This spiral segment starts at p_1 at distance $A' = A(e^{2\pi\cot\alpha} - 1)$ from its center p_0. Since p_2 and p_1 form an angle α at p_0, the distance from p_2 to p_0 equals $A' e^{\alpha\cot\alpha}$. Thus, the curce length from p_1 to p_2 is given by $l_2' = \frac{A'}{\cos\alpha}(e^{\alpha\cot\alpha} - 1) = \frac{A}{\cos\alpha}(e^{2\pi\cot\alpha} - 1)(e^{\alpha\cot\alpha} - 1)$. Consequently, the overall curve length l_2 from p_0 to p_2 equals

$$l_2 \;=\; l_1 + l_2' \;=\; \frac{A}{\cos\alpha}(e^{2\pi\cot\alpha} - 1)e^{\alpha\cot\alpha} \tag{5.16}$$

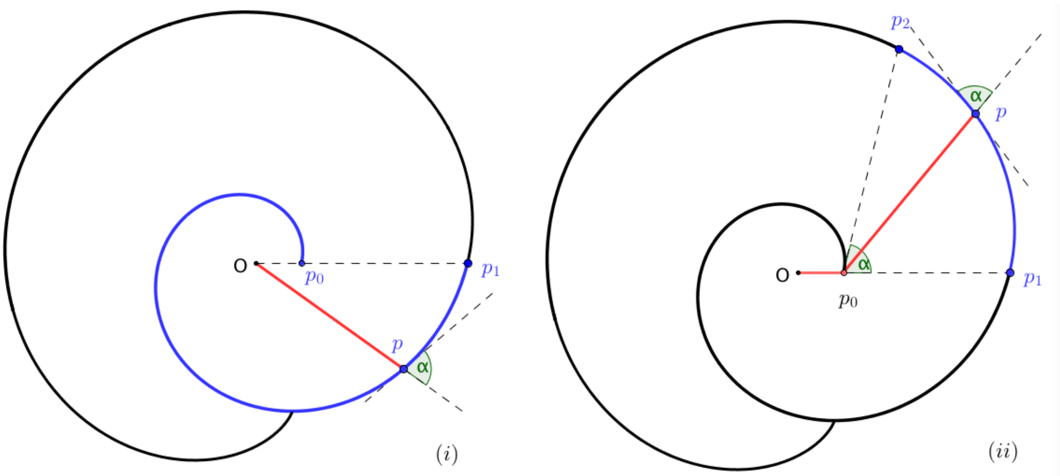

Figure 5.17: The barrier curve starts with two parts of logarithmic spirals of excentricity α, centered at 0 and p_0, respectively.

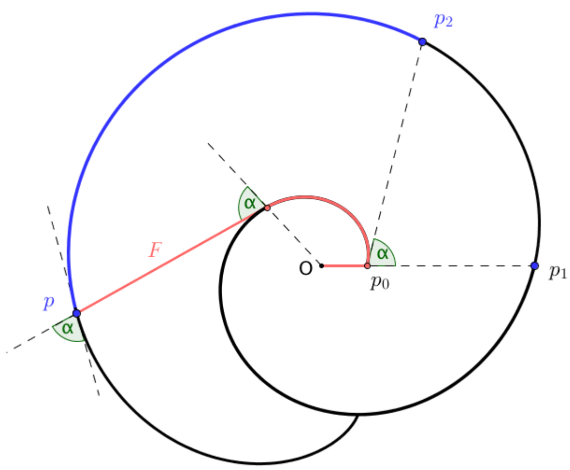

Figure 5.18: From point p_2 on the barrier curve results from a wrapping around the already constructed barrier. The last segment, *free string* F, of the shortest path from the fire source to the current barrier point p shrinks, by wrapping, and simultaneously grows by $\cos\alpha$. The barrier curve starting from p_2 is no longer a logarithmic spiral. The strategy will be successful if F shrinks to zero.

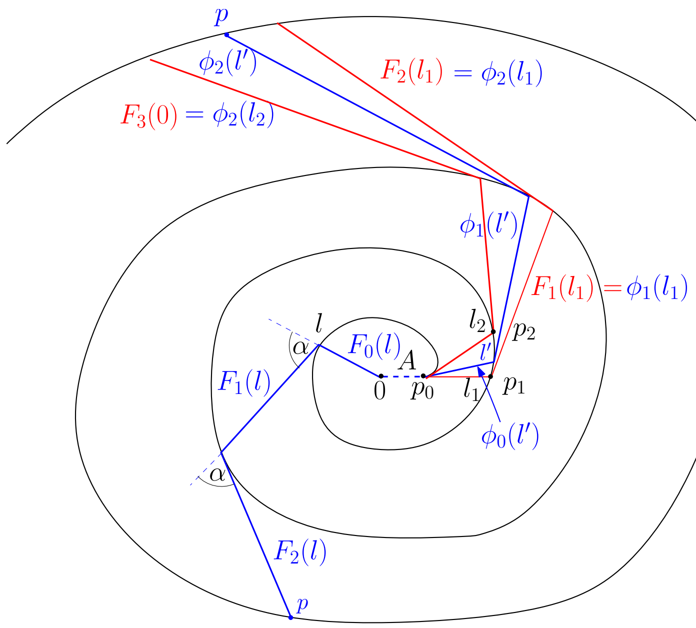

Figure 5.19: Repeatedly constructing backwards tangents may end in 0 or in p_0. This way, two types of linkages are defined.

From point p_2 on, the geodesic shortest path from 0 to the fighter's current position, p, starts wrapping around the existing spiral part of the curve, beginning at p_0; see Figure 5.18. The last segment of this path is tangent to the previous round of the curve. As mentioned in the Introduction, we shall endeavor to determine its length, F, because the fire will be contained if and only if F ever attains the value 0.

One could think of this tangent as a string (named the *free string*) at whose endpoint, p, a pencil is attached that draws the barrier curve. But unlike an involute, here the string is not normal to the outer layer. Rather, its extension beyond p forms an angle α with the barrier's tangent at p. This causes the string to grow in length by $\cos\alpha$ for each unit drawn. At the same time, the inner part of the string gets wrapped around the previous coil of the barrier. It is this interplay between growing and wrapping that we will investigate below.

One can show that after p_2 the barrier curve is no longer segment of a logarithmic spiral. But to give a closed form representation for the second round, leave alone for subsequent rounds recursively constructed, seems to be out of reach.

At the end of this subsection, let us recall the following fact. As the fighter is building the barrier at speed $v = 1/\cos\alpha$, the fire is coming after her at unit speed along the outside of the barrier, as indicated in Figure 5.15. Thus, each barrier point p is caught by fire twice, once from the inside, when the fighter passes through p, and a second time from the outside, if the fire is not stopped before.

5.6.2 Structural properties

In this subsection we assume that the fighter has built quite a few rounds of the barrier curve without yet containing the fire. That the first two rounds of the curve involve two different spiral segments, around 0 and around p_0, carries over to subsequent layers. The structure of the curve can be described as follows. Let l_1 and l_2 denote the curve lengths from p_0 to p_1 and p_2, respectively, as in Equations 5.15 and 5.16. For $l \in [0, l_1]$ let $F_0(l)$ denote the segment connecting 0 to the point of curve length l; see the sketch given in Figure 5.19. At the endpoint of $F_0(l)$ we construct the tangent and extend it until it hits the next layer of the curve, creating a segment $F_1(l)$, and so on. This construction gives rise to a "linkage" connecting adjacent layers of the curve. Each edge of the linkage is turned counterclockwise by α with respect to its predecessor. The outermost edge of a linkage is the free string mentioned above. As parameter l increases from 0 to l_1, edge $F_0(l)$, and the whole linkage, rotate counterclockwise. While $F_0(0)$ equals the line segment from the origin to p_0, edge $F_0(l_1)$ equals segment $0p_1$.

Analogously, let $l' \in [l_1, l_2]$, and let $\phi_0(l')$ denote the segment from p_0 to the point at curve length l' from p_1. This segment can be extended into a linkage in the same way. We observe that

$$F_{j+1}(l_1) = \phi_{j+1}(l_1) \tag{5.17}$$
$$F_{j+1}(0) = \phi_j(l_2) \tag{5.18}$$

hold (but initially, we have $F_0(l) = A + \cos(\alpha)\, l$ and $\phi_0(l') = \cos(\alpha)\, l'$, so that $F_0(l_1) \neq \phi_0(l_1)$). Clearly, each point on the curve can be reached by a unique linkage, as tangents can be constructed backwards. We refer to the two types of linkages by F-type and ϕ-type. As Figure 5.19 illustrates, points of the same linkage type form alternating intervals along the barrier curve. If p's linkage is of F-type then p is uniquely determined by the index $j \geq 0$ and parameter $l \in [0, l_1]$ such that p is the outer endpoint of edge $F_j(l)$.

Now we will derive two structural properties of F-linkages on which our analysis will be based; analogous facts hold for ϕ-linkages, too. To this end, let $L_j(l)$ denote the length of the barrier curve from p_0 to the outer endpoint of edge $F_j(l)$, and let $F_j(l)$ also denote its length of edge $F_j(l)$.

Lemma 60 *We have $L_{j-1}(l) + F_j(l) = \cos\alpha\, L_j(l)$.*

Proof. Both, fire and fire fighter, reach the endpoint of $F_j(l)$ at the same time. The fire has travelled a geodesic distance of $L_{j-1}(l) + F_j(l)$ at unit speed, the fighter a distance of $L_j(l)$ at speed $1/\cos\alpha$. □

Equivalently, one could argue that the free string grows by $\cos\alpha$ for each barrier unit built; thus, the total string length (wrapped plus free) of $L_{j-1}(l) + F_j(l)$ must be equal to $\cos\alpha\, L_j(l)$.

The second property is related to the wrapping of the free string. Intuitively, it says that if we turn an F-linkage, the speed of each edge's endpoint is proportional to its length (with the same proportionality constant for all j depending on turning speed).

Lemma 61 *As functions in l, L_j and F_j satisfy the following equation.*

$$\frac{L'_{j-1}(l)}{L'_j(l)} = \frac{F_{j-1}(l)}{F_j(l)}.$$

First, we will derive Lemma 61 from two general facts on smooth curves stated in Lemma 62 and Lemma 63.

Lemma 62 *Suppose a string of length F is tangent to a point t on some smooth curve C. Now the end of the string moves a distance of ϵ in the direction of α, as shown in Figure 5.20. Then for the curve length $C_t^{t_\epsilon}$ between t and the new tangent point, t_ϵ, we have*

$$\lim_{\epsilon \to 0} \frac{C_t^{t_\epsilon}}{\epsilon} = \frac{\sin\alpha\ r}{F}$$

where r denotes the radius of the osculating circle at t.

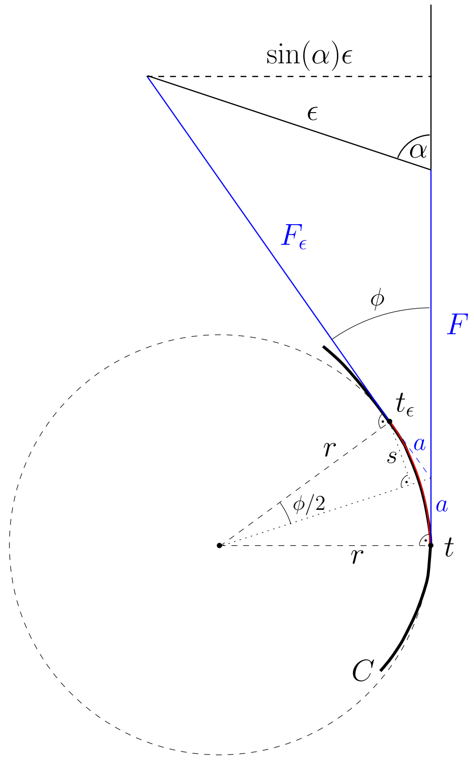

Figure 5.20: A string wrapping around a curve.

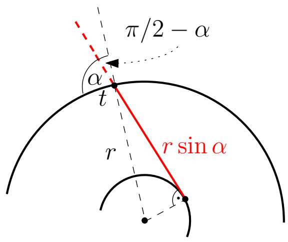

Figure 5.21: Intersection of turned normals.

This fact is quite intuitive. The more perpendicular the motion of the string's endpoint, and the larger the radius of curvature, the more string gets wrapped. But if the string is very long, the effect of the motion decreases; see also Figure 5.20.

The center of the osculating circle at t is known to be the limit of the intersections of the normals of all points near t with the normal at t. Lemma 63 shows what happens if, instead of the normals, we consider the lines turned by the angle $\pi/2 - \alpha$.

Lemma 63 *Let t be a point on a smooth curve C, whose osculating circle at t is of radius r. Consider the lines L_s resulting from turning the normal at points s by an angle of $\pi/2-\alpha$. Then their limit intersection point with L_t has distance* $\sin\alpha\, r$ *to t.*

A simple example is shown in Figure 5.21 for the case where curve C itself is a circle. Now we can prove Lemma 61.

Proof.[Proof of Lemma 61] By Lemma 62, applied to the inner point t of edge $F_j(L_j)$, we have

$$\frac{L'_{j-1}(L_j)}{L'_j(L_j)} = L'_{j-1}(L_j) = \frac{\sin\alpha\ r}{F_j(L_j)}.$$

Lemma 63 implies that $\sin\alpha\ r$ equals the distance between t and the limit intersection point of the normals turned by $\pi/2-\alpha$ near t. But for the barrier curve generated by strategy FF, these turned normals are the tangents to the previous coil, so that $\sin\alpha\ r = F_{j-1}(L_j)$ holds. As we substitute variable L_j with $L_j(l)$, the derivatives of the inner functions cancel out and we obtain Lemma 61. □

The proofs of Lemma 63 and 62 are rather straightforward. We include them for completeness.

Proof.[Proof of Lemma 62] Using the notations in Figure 5.20, the following hold. From $r\ \sin(\phi/2) = s = a\ \cos(\phi/2)$ we obtain $a = r\ \tan(\phi/2)$. For short, let $c := C_t^{t_\epsilon}$. By l'Hospital's rule,

$$\frac{c}{2a} \ = \ \frac{r\,\phi}{2r\ \tan(\phi/2)} \ \approx\ \cos^2(\phi/2)\ \to\ 1$$

as ϵ, hence ϕ, go to 0. Thus, $2a$ is a good approximation of $c = C_t^{t_\epsilon}$. By the law of sines,

$$\frac{\epsilon\ \sin(\alpha)}{\sin(\phi)} \ = \ \frac{F_\epsilon + a}{\sin(\pi/2)},$$

hence

$$\frac{\sin(\phi)}{\epsilon} \ = \ \frac{\sin(\alpha)}{F_\epsilon + a} \ \to\ \frac{\sin(\alpha)}{F}$$

This implies $\sin(\phi/2)/\epsilon\ \to\ \sin(\alpha)/(2F)$, and we conclude

$$\frac{C_t^{t_\epsilon}}{\epsilon} \ = \ \frac{c}{2a}\,\frac{2a}{\epsilon} \approx \frac{2r\ \tan(\phi/2)}{\epsilon} = \frac{2r\ \sin(\phi/2)}{\epsilon\ \cos(\phi/2)} \ \to\ \frac{r\sin(\alpha)}{F}.$$

□

Proof.[Proof of Lemma 63] Let us assume that C is locally parameterized by $Y = f(X)$ and that $t = (x_0, f(x_0))$. Then the tangent in t is

$$Y \ = \ f'(x_0)X \ - \ f'(x_0)x_0 \ + \ f(x_0),$$

and line L_t, the tangent turned counterclockwise by α, is given by

$$Y \ = \ \tan(\arctan(f'(x_0)) + \alpha)X \ - \ \tan(\arctan(f'(x_0)) + \alpha)x_0 \ + \ f(x_0).$$

Now let (v, w) denote the point of intersection of L_t and L_s, where $s = (x_0 + \epsilon, f(x_0 + \epsilon))$. Equating the two line equations we obtain

$$\big(h(x_0+\epsilon) - h(x_0)\big)\, v \ = \ g(x_0+\epsilon) - g(x_0) + f(x_0) - f(x_0+\epsilon)$$

where

$$h(x) \ := \ \tan(\arctan(f'(x)) + \alpha) \ \text{ and } \ g(x) \ := \ h(x)x$$

After dividing by ϵ and taking limits, we have

$$h'(x_0)\, v_0 = g'(x_0) - f'(x_0) = h'(x_0)\, x_0 + h(x_0) - f'(x_0),$$

which results in

$$\begin{aligned} v_0 \ &= \ x_0 + \frac{h(x_0) - f'(x_0)}{h'(x_0)} \\ w_0 \ &= \ h(x_0)\, v_0 - g(x_0) + f(x_0) \\ &= \ f(x_0) + \frac{h^2(x_0) - h(x_0)f'(x_0)}{h'(x_0)}. \end{aligned}$$

In other words,

$$
\begin{aligned}
(v_0, w_0) - (x_0, f(x_0)) &= \frac{h(x_0) - f'(x_0)}{h'(x_0)} (1, h(x_0)) \\
|(v_0, w_0) - (x_0, f(x_0))| &= |\frac{h(x_0) - f'(x_0)}{h'(x_0)}| \sqrt{1 + h^2(x_0)}
\end{aligned}
$$

Using the addition formula for tan,

$$
h(x) = \tan(\arctan(f'(x)) + \alpha) = \frac{f'(x) + \tan(\alpha)}{1 - f'(x)\tan(\alpha)},
$$

we obtain

$$
h(x_0) - f'(x_0) = \frac{1 + (f'(x_0))^2 + \tan(\alpha)}{1 - f'(x_0)\tan(\alpha)}.
$$

and

$$
1 + h^2(x_0) = \frac{(1 + (f'(x_0))^2)(1 + \tan^2(\alpha))}{(1 - f'(x_0)\tan(\alpha))^2}.
$$

Moreover,

$$
h'(x_0) = \frac{f''(x)(1 + \tan^2(\alpha))}{(1 - f'(x_0)\tan(\alpha))^2}.
$$

Putting expressions together we obtain

$$
|(v_0, w_0) - (x_0, f(x_0))| = |\frac{\left(1 + (f'(x_0))^2\right)^{3/2}}{f''(x_0)}| \frac{\tan(\alpha)}{\sqrt{1 + \tan^2(\alpha)}}.
$$

The first term is known to be the radius of the osculating circle, r, and the second equals $\sin(\alpha)$. □

5.6.3 Differential equations

In this section we turn the structural properties observed in Subsection 5.6.2 into differential equations.

By multiplication, Lemma 61 generalizes to non-consecutive edges. Thus,

$$
\frac{F_j(l)}{F_0(l)} = \frac{L_j'(l)}{l'} = L_j'(l) \tag{5.19}
$$

holds. On the other hand, taking the derivative of the formula in Lemma 60 leads to

$$
F_j'(l) + L_{j-1}'(l) = \cos\alpha\, L_j'(l). \tag{5.20}
$$

We substitute in 5.20 both $L_j'(l)$ and $L_{j-1}'(l)$ by the expressions we get from 5.19 and obtain a linear differential equation for $F_j(l)$,

$$
F_j'(l) - \frac{\cos(\alpha)}{F_0(l)} F_j(l) = -\frac{F_{j-1}(l)}{F_0(l)}.
$$

The textbook solution for $y'(x) + f(x)y(x) = g(x)$ is

$$
y(x) = \exp(-a(x)) \left(\int g(t) \exp(a(t))\, \mathrm{d}t + \kappa \right),
$$

where $a = \int f$ and κ denotes a constant that can be chosen arbitrarily. In our case,

$$a(l) \;=\; \int -\frac{\cos(\alpha)}{A + \cos(\alpha)\, l} \;=\; -\ln(F_0(l))$$

because of $F_0(l) = A + \cos(\alpha)\, l$, and we obtain

$$F_j(l) \;=\; F_0(l)\Big(\kappa_j \;-\; \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big). \tag{5.21}$$

Next, we consider a linkage of ϕ-type, for parameters $l \in [l_1, l_2]$, and obtain analogously

$$\phi_j(l) \;=\; \phi_0(l)\Big(\lambda_j \;-\; \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)}\,\mathrm{d}t\Big). \tag{5.22}$$

Now we determine the constants κ_j, λ_j such that the solutions 5.21 and 5.22 describe a contiguous curve. To this end, we must satisfy conditions 5.17 and 5.18.

We define $\kappa_0 := 1$ and

$$\kappa_{j+1} := \frac{\phi_j(l_2)}{F_0(0)} \;+\; \int \frac{F_j(t)}{F_0^2(t)}\mathrm{d}t|_{l=0}$$

so that 5.21 becomes

$$F_{j+1}(l) \;=\; F_0(l)\Big(\frac{\phi_j(l_2)}{F_0(0)} \;-\; \int_0^l \frac{F_j(t)}{F_0^2(t)}\,\mathrm{d}t\Big),$$

which, for $l = 0$, yields $F_{j+1}(0) = \phi_j(l_2)$ (condition 5.18).

Similarly, we set $\lambda_0 := 1$ and

$$\lambda_{j+1} := \frac{F_{j+1}(l_1)}{\phi_0(l_1)} \;+\; \int \frac{\phi_j(t)}{\phi_0^2(t)}\mathrm{d}t|_{l=l_1}$$

so that 5.22 becomes

$$\phi_{j+1}(l) \;=\; \phi_0(l)\Big(\frac{F_{j+1}(l_1)}{\phi_0(l_1)} \;-\; \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)}\,\mathrm{d}t\Big),$$

and for $l = l_1$ we get $F_{j+1}(l_1) = \phi_{j+1}(l_1)$ (condition 5.17).

For simplicity, let us write

$$G_j(l) \;:=\; \frac{F_j(l)}{F_0(l)} \quad\text{and}\quad \chi_j(l) \;:=\; \frac{\phi_j(l)}{\phi_0(l)}, \tag{5.23}$$

which leads to

$$G_{j+1}(l) \;=\; \frac{\phi_0(l_2)}{F_0(0)}\,\chi_j(l_2) \;-\; \int_0^l \frac{G_j(t)}{F_0(t)}\,\mathrm{d}t \tag{5.24}$$

$$\chi_{j+1}(l) \;=\; \frac{F_0(l_1)}{\phi_0(l_1)}\,G_{j+1}(l_1) \;-\; \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)}\,\mathrm{d}t. \tag{5.25}$$

Now we can make a useful observation. In order to find out if the fire fighter is ever successful, we need to check only the values of $F_j(l)$ at the end of each round.

Lemma 64 *The curve encloses the fire if and only if there exists an index j such that $F_j(l_1) \leq 0$ holds.*

Proof. The free string shrinks to zero if and only if there exist an index j and argument l such that $F_j(l) \leq 0$ or $\phi_j(l) \leq 0$. Clearly, G_j and F_j have identical signs, as well as χ_j and ϕ_j do. Suppose that $G_j > 0$ and $G_{j+1}(l) = 0$, for some j and some $l \in [0, l_1]$. By 5.24, function G_{j+1} is decreasing, therefore $G_{j+1}(l_1) \leq 0$. Now assume that $G_i > 0$ holds for all i, and that we have $\chi_{j-1} > 0$ and $\chi_j(l) = 0$ for some j and some $l \in [l_1, l_2]$. By 5.25 this implies $\chi_j(l_2) \leq 0$, and from 5.24 we conclude $G_{j+1} \leq 0$, in particular $G_{j+1}(l_1) \leq 0$. □