Arrangement of Hyperplanes (Chapter 6.1 and Chapter 6.3)

For a set H of hyperplanes in $\mathbb{R}^d$, the arrangement of H is a partition of $\mathbb{R}^d$ into relatively open convex faces.

- 0-faces called vertices
- 1-faces called edges
- $(d-1)$-faces called facets
- d-faces called cells.

Faces in the arrangement

- The cells are the connected components of $\mathbb{R}^d \setminus \bigcup H$.
- The facets are obtained from the $(d-1)$-dimensional arrangements induced in the hyperplanes of H by their intersections with the other hyperplanes
  - For each $h \in H$, take the connected components of $h \setminus \bigcup_{h' \in H, h' \neq h} h'$.
- k-faces are obtained from every possible k-flat L defined as the intersection of some $d-k$ hyperplanes of H
  - The k-faces of the arrangement lying within L are the connected components of $L \setminus (H \setminus H_L)$, where $H_L = \{h \in H \mid L \subseteq h\}$

Sign Vectors:
A face F of the arrangement of H can be described by its sign vectors

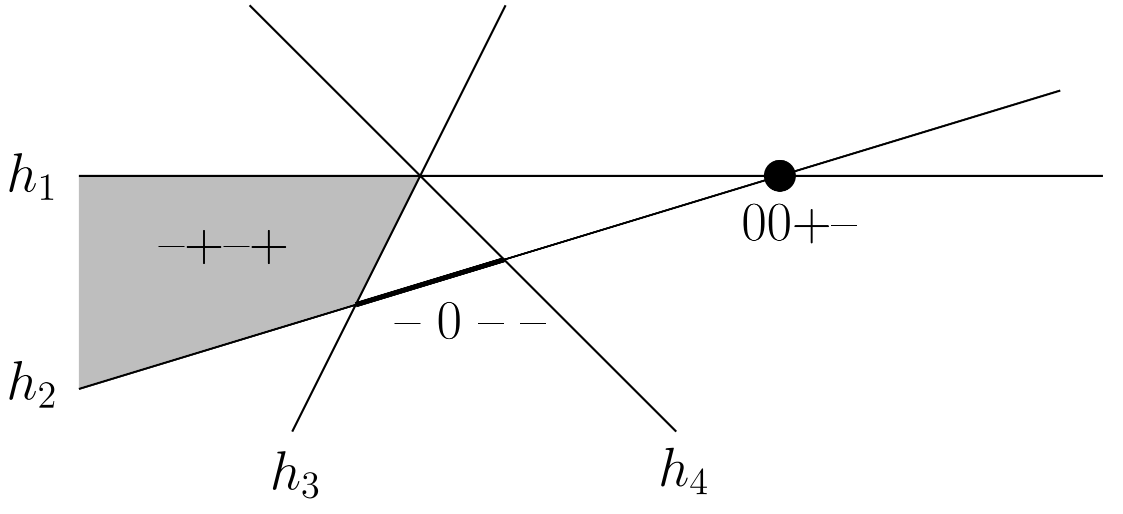
- Fix the orientation of each hyperplane
  - Each $h \in H$ partitions $\mathbb{R}^d$ into three regions: h itself and the two open half-spaces determined by it.
  - Choose one of these open half-spaces as positive and denote it by $h^{\oplus}$, and we let the other one be negative and dnote it by $h^{\ominus}$,
- The *sign vector* of F is defined as $\sigma(F) = (\sigma_h \mid h \in H)$ where

$$\sigma_h = \begin{cases} +1 \text{ if } F \subseteq h^{\oplus}, \\ 0 \text{ if } F \subseteq h, \\ -1 \text{ if } F \subseteq h^{\ominus}. \end{cases}$$

The face F is determined by its sign vector, since we have

$$F = \bigcap_{h \in H} h^{\sigma_h},$$

where $h_0 = h$, $h^1 = h^{\oplus}$, and $h^{-1} = h^{\ominus}$.



Not all possible sign vectors corresponds to nonempty faces

- For n lines, there 3^n sign vectors but only $O(n^2)$ faces.

Counting the cells in a hyperplane arrangement

- General Position
  - The intersection of every k hyperplanes is $(d-k)$-dimensional, $k = 2, 3, \ldots, d+1$.
  - If $H \geq d+1$, then it suffices to require that every d hyperplanes intersect at a single point, and no $d+1$ hyperplane have a common point.
- If H is in general position, the arrangement of H is called *simple*
- Every d-tuple of hyperplanes in a simple arrangement determines exactly one vertex, so a simple arrangement of n hyperplanes has exactly $\binom{n}{d}$ vertices.
- The number of cells will be shown to be $O(n^d)$ for d fixed.

Proposition

The number of cell (d-faces) in a simple arrangement of n hyperplane in $\mathbb{R}^d$ equals

$$\phi_d(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}$$

First proof

- Proceed by induction on the dimension d and the number of hyperplanes n.
- For $d = 1$
  - We have a line and n points in it
  - These divide the line into $n+1$ one-dimensional pieces, and the statement holds.
- For $n = 0$ and $d \geq 1$, it trivially holds.
- Suppose we are in dimension d, we have $n-1$ hyperplanes, and we insert another one h
- By the inductive hypothesis, the $n-1$ previous hyperplanes divide the newly inserted hyperplane h into $\phi_{d-1}(n-1)$ cells
- Each such $(d-1)$-dimensional caell within h partitions one d-dimensional cell into exactly two cells.
- The total increase in the number of cells caused by inserting h is $\phi_{d-1}(n-1)$, so
$$\phi_d(n) = \phi_d(n-1) + \phi_{d-1}(n-1).$$
- Together with the intial condition (for $d = 1$ and $n = 0$), it remains to check the formula satisfies the recurrence
$$\begin{aligned}\phi_d(n-1) + \phi_{d-1}(n-1) &= \tbinom{n-1}{0} + [\tbinom{n-1}{1} + \tbinom{n-1}{0}] \\ &\quad + [\tbinom{n-1}{2} + \tbinom{n-1}{1}] + \cdots [\tbinom{n-1}{d} + \tbinom{n-1}{d-1}] \\ &= \tbinom{n-1}{0} + \tbinom{n}{1} + \tbinom{n}{2} + \cdots + \tbinom{n}{d} = \phi_d(n).\end{aligned}$$

second proof

- Proceed by induction on d, the case $d = 0$ being trivial.
- Let H be a set of n hyperplanes in $\mathbb{R}^d$ in general posistion
  - Assume no hyperplanes of H is hortizontal
  - Assume no two vertices of the arrangement have the same vertical-level (x_d-coordinate)
- Let g be an auxiliary horizonal hyperplane lying below all the vertices
- A cell of the arrangement of H is
  - bounded from below, and in this case it has a unique vertex,
  - or is not bounded from below, and then it intersects g
- The number of cells of the former type is the same as the number of vertices, which is $\binom{n}{d}$.
- The cells of the latter type correspond to the cells in the $(d-1)$-dimensional arrangement induced within g by the hyperplanes of H, and their number if thues $\phi_{d-1}(n)$.

Level of a point

For a set H of hyperplanes in $\mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, the *level* of x with respect to H is the number of hyperplanes in H lying strictly below x.

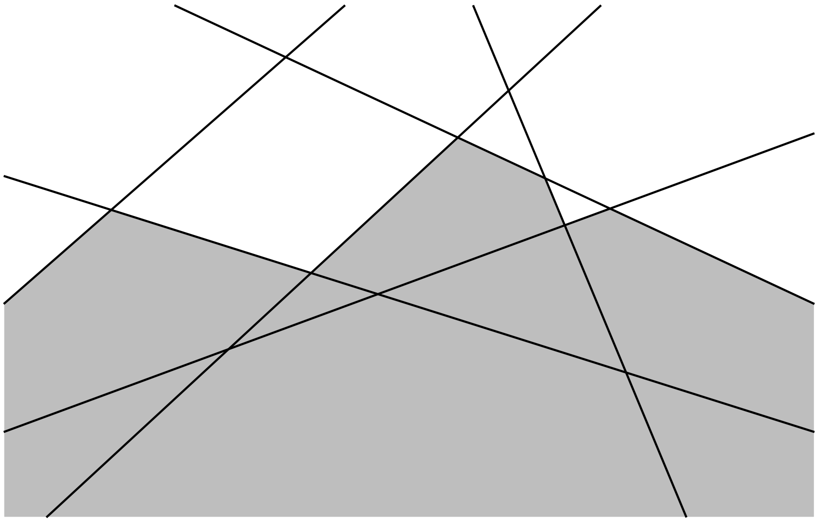
k-level

For a set H of n hyperplanes in $\mathbb{R}^d$, the k-level of the arrangement of H is the closure of facets in the arrangement whose interior points have a level of k with respect to H.

- The size of the k-level is counted by its vertices
- For $d = 2$, its size is $\Omega(n2^{\sqrt{\log k}})$ and $O(nk^{1/3})$
- For $d = 3$, its size is $\Omega(nk2^{\sqrt{\log k}})$ amd $O(nk^{3/2})$.
- The k-level is dual to the k-set

At most k-levels For a set H of n hyperplanes in $\mathbb{R}^d$, the at most k-levels, denoted by $\leq k$-level, is the collection of i-level for $0 \leq i \leq k$.

- its size is counted by the number of vertices.



$\leq$ 2-level

0-level has $O(n^{\lfloor d/2 \rfloor})$ vertices

- The vertices of the 0-level are the vertices of the cell lying below all the hyperplanes
- This cell is the intersection of at most n half-space.

Clarkson's theorem on levels

The total number of vertices of level at most k in an arrangement of n hyperplanes in $\mathbb{R}^d$ is at most

$$O(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$$

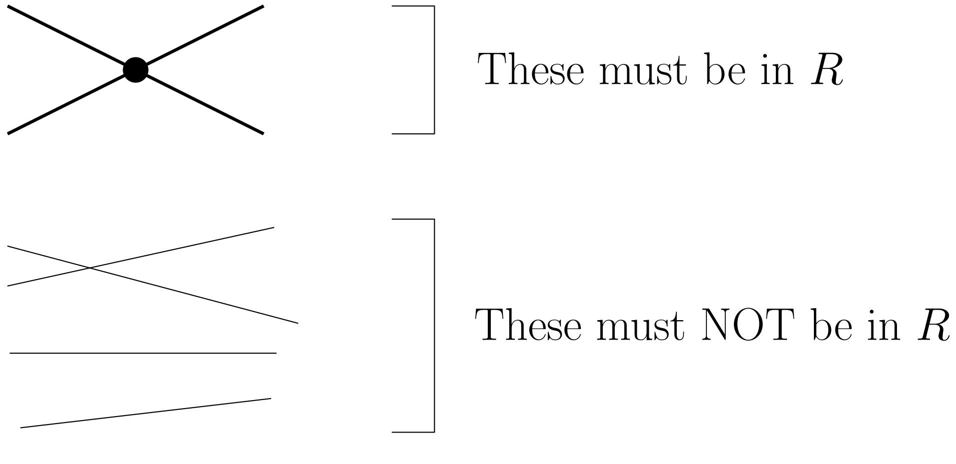
The lower bound for the number of vertices of level at most k is

$$\Omega(n^{\lfloor d/2 \rfloor}k^{\lceil d/2 \rceil})$$

- Consider a set of $\frac{n}{k}$ hyperplanes such taht the lower unbounded cell in their arrangement is a convex polyhedron with $\Omega(\binom{n}{k}^{\lfloor d/2 \rfloor})$ vertices
- Replace each of the hyperplanes by k very close parallel hyperplanes.
- Each vertex of level 0 in the original arrangement gives rise to $\Omega(k^d)$ vertices of level at most k in the new arrangement

Proof of Clarkson's Theorem for $d = 2$

- Let H be a set of n lines in general position
- Let p denote a certain suitable number in the interval $(0, 1)$
- Imagine a random experiment
  - Choose a subset $R \subseteq H$ at random, by including each line $h \in H$ into R with probability p
    - the choices are independent for distinct lines h.
  - Consider the arrangement of R, and let $f(R)$ denote the number of vertices of level 0 in the arrangement of R.
  - Since R is randomly chosen, f is a random variable.
  - Estimate the expectation of f, denoted by $E[f]$.
  - For any specific set R, we have $f(R) \leq |R|$, so $E[f] \leq E[|R|] = pn$.
  - Bound $E[f]$ from below
    - For each vertex v of the arrangement of H, we define an event A_v meaning "v becomes one of the vertices of level 0 in the arrangement of R."
    - That is, A_v occurs if v contribute 1 to the value of f
    - A_v occurs if and only if the following two conditions are satisfied:
      - Both lines determining v lie in R
      - None of the lines of H lying below v falls into R



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$$\text{Prob}[A_v] = p^2(1 - p)^{l(v)},$$

where $l(v)$ denotes the level of the vertex v

- Let V be the set of all vertices of the arrangement of H, and let $V_{\leq k} \subseteq V$ be the set of vertices of level at most k.

$$E[f] = \sum_{v \in V} \text{Prob}[A_v] \geq \sum_{v \in V_{\leq k}} \text{Prob}[A_v]$$

$$= \sum_{v \in V_{\leq k}} p^2(1-p)^{l(v)} \geq \sum_{v \in V_{\leq k}} p^2(1-p)^k = |V_{\leq k}| \cdot p^2(1-p)^k.$$

- Since $np \geq E[f] \geq |V_{\leq k}| \cdot p^2(1-p)^k$,

$$|V_{\leq k}| \leq \frac{n}{p(1-p)^k}.$$

- Choose the number p to minimize the right hand side
  - A convenient value is $p = \frac{1}{k+1}$
  - Since $\left(1 - \frac{1}{k+1}\right)^k \geq e^{-1} > \frac{1}{3}$ for all $k \geq 1$,

$$|V_{\leq k}| \leq 3(k+1)n.$$

Proor for an arbitrary dimensions

- Define an integer parameter r and choose a random r-element subset $R \subseteq H$, with all $\binom{n}{r}$ subsets being equally probable.
- Define $f(R)$ as the number of vertices of level 0 with respect to R, and estimate $E[f]$ in two ways (from up and below).
- Since $f(R) = O(r^{\lfloor d/2 \rfloor})$ for all R,

$$E[f] = O(r^{\lfloor d/2 \rfloor}).$$

- Let V be the set of all vertices in the arrangement of H, $V_{\leq k}$ be the set of vertices in V whose level with respect to H is at most k, and A_v be the event "v is a vertex of level 0 with respect to R."
- The conditions for A_v are
  - All the d hyperplane defining the vertex v fall in R.
  - None of the hyperplane of H lying below v fall in R.

- If $l = l(v)$ is the level of v, then

$$\text{Prob}[A_v] = \frac{\binom{n-d-l}{r-d}}{\binom{n}{r}}.$$

- Let $P(l)$ denote $\frac{\binom{n-d-l}{r-d}}{\binom{n}{r}}$.
  - $P(l)$ is a decreasing function.
- Therefore,

$$E[f] = \sum_{v \in V} \text{Prob}[A_v] \geq V_{\leq k} \cdot P(k).$$

- Combining with $E[f] = O(r^{\lfloor d/2 \rfloor})$, we obtain

$$|V_{\leq k}| \leq \frac{O(r^{\lfloor d/2 \rfloor})}{P(k)}.$$

- Set r be $\lfloor \frac{n}{k+1} \rfloor$.
  - as inspired by the case for $d = 2$, where $pn = \frac{n}{k+1}$.
- We will prove latter that If $1 \leq k < \frac{n}{2d} - 1$,

$$P(k) \geq c_d (k+1)^{-d}$$

  for a suitable $c_d > 0$ depending only on d.
- Combining $|V_{\leq k}| \leq \frac{O(r^{\lfloor d/2 \rfloor})}{P(k)}$, $P(k) \geq c_d(k+1)^{-d}$, and $r = \lfloor \frac{n}{k+1} \rfloor$, we have

$$|V_{\leq k}| \leq O(r^{\lfloor d/2 \rfloor})(k+1)^d = O\left(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil}\right)$$

- For $k \geq \frac{n}{2d}$, the bound claimed by this theorem is $O(n^d)$ and thus trivial.
- For $k = 0$, the bound is $O(n^{\lfloor d/2 \rfloor})$ and alreay known.

Lemma Suppose that $1 \leq k \leq \frac{n}{2d} - 1$, which implies $2d \leq r \leq \frac{n}{2}$. Then

$$P(k) \geq c_d (k+1)^{-d}$$

for a suitable $c_d > 0$ depending only on d.

$$P(k) = \frac{\binom{n-d-k}{r-d}}{\binom{n}{r}}$$

$$= \frac{(n-d-k)(n-d-k-1)\cdots(n-k-r+1)}{n(n-1)\cdots(n-r+1)} \cdot r(r-1)\cdots(r-d+1)$$

$$= \frac{r(r-1)\cdots(r-d+1)}{n(n-1)\cdots(n-d+1)} \cdot \frac{n-d-k}{n-d} \cdot \frac{n-d-k-1}{n-d-1} \cdots \frac{n-k-r-1+1}{n-r+1}$$

$$\geq \big(\frac{r}{2n}\big)^d \big(1 - \frac{k}{n-d}\big)\big(1 - \frac{k}{n-d-1}\big) \cdots \big(1 - \frac{k}{n-r+1}\big)$$

$$\geq \big(\frac{r}{2n}\big)^d \big(1 - \frac{k}{n-r+1}\big)^r$$

- Since $k < \frac{n}{2}$,
  - $\frac{r}{n} \geq (\frac{n}{k+1} - 1)/n \geq \frac{1}{2(k+1)}$. (recall $r = \lfloor \frac{n}{k+1} \rfloor$.)
  - $1 - \frac{k}{n-r+1} \geq 1 - \frac{2k}{n}$.
- Since $k \leq \frac{n}{4}$, we can use the inequality $1 - x \geq e^{-2x}$
- Finally,

$$P(k) \geq \big(\frac{r}{dn}\big)^d \big(1 - \frac{2k}{n}\big)^r \geq \big(\frac{1}{(k+1)\cdot d}\big)^d e^{-4kr/n} \geq c_d (k+1)^{-d}$$