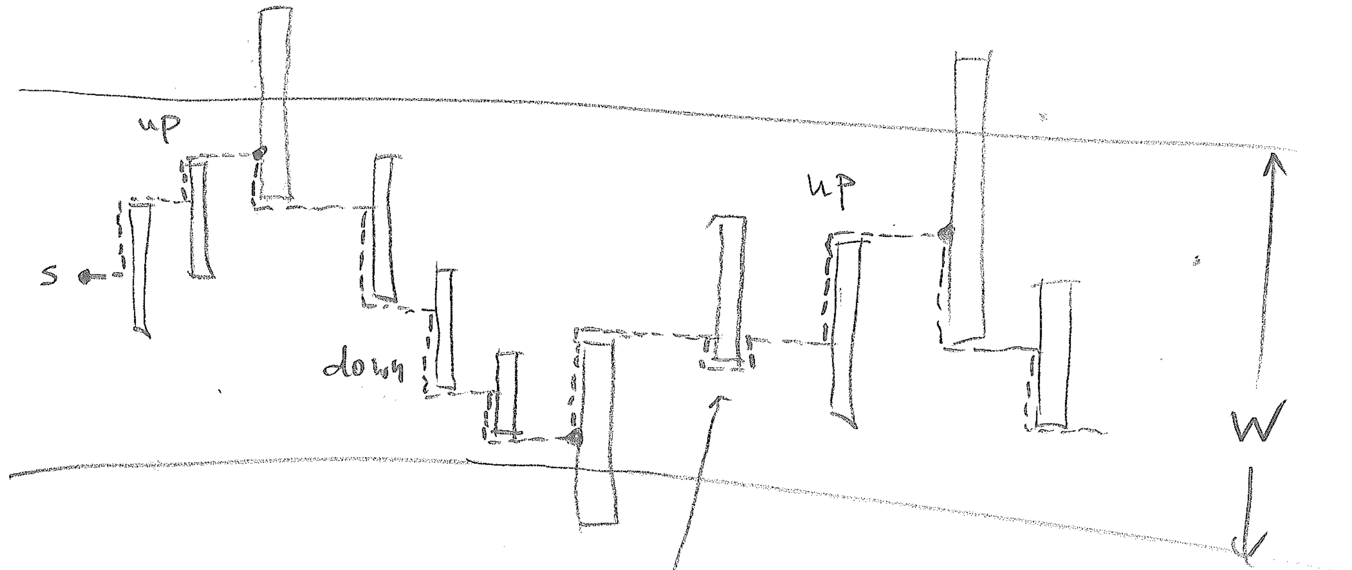
Theorem (Blum, Raghavan, Schieber '91)
There exists an $O(\sqrt{n})$ competitive strategy for finding a line at known position amidst n axis-parallel rectangular obstacles.

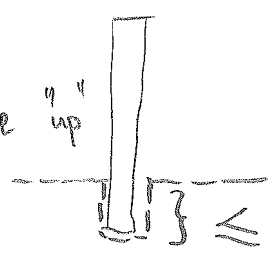
Proof Ideas: (1) "go to upper corner" not too bad if alternated with "go to lower corner"

(2) excursions not too bad if short enough

Consider a horizontal strip of width W.
While moving towards t, robot crosses strip $\sqrt{n}$ many times alternating directions "up" and "down"

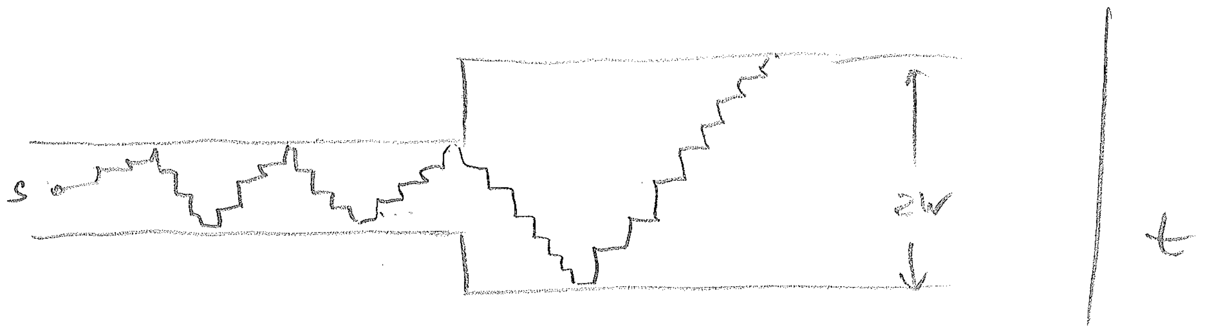


exceptional excursions : same in mode "down".

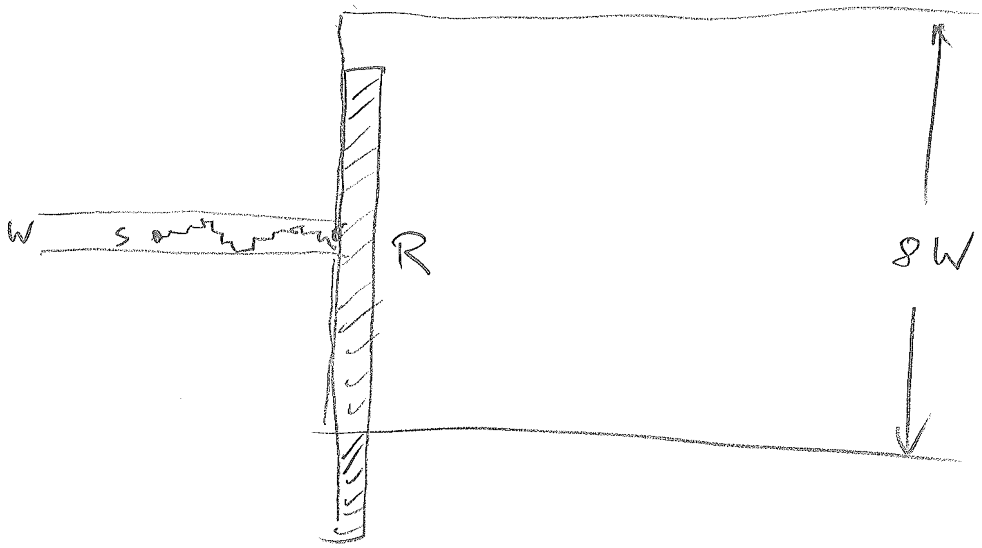


$\} \leq \tau = \frac{W}{\sqrt{n}}$

If target line t not found after $\sqrt{n}$ crossings of strip double strip width $W \leftarrow 2W$

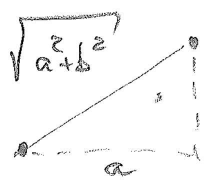


if obstacle R blocks strip:
double strip width, until one end of R contained in strip



<u>Analysis</u> (no strip size doubling, first)

(uses: $L_2 \le L_1 \le \sqrt{2}\, L_2$)
L_1 = Manhattan distance



<u>robot's path in strip of width W</u> (upper bounds)

horizontal distance: n

$\sqrt{n}$ times "up" and "down" across strip $\sqrt{n}\, W$

excursions of length $2T = 2\frac{W}{\sqrt{n}}$: $n\, 2\frac{W}{\sqrt{n}}$

at most n, because of unit circle assumption

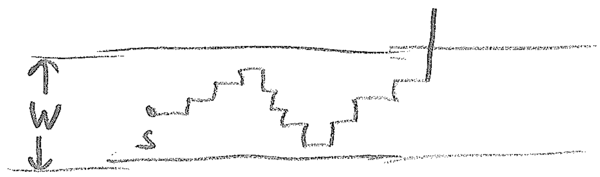
$n + 3\sqrt{n}\, W$

Shortest path: (lower bounds)

horizontal distance n

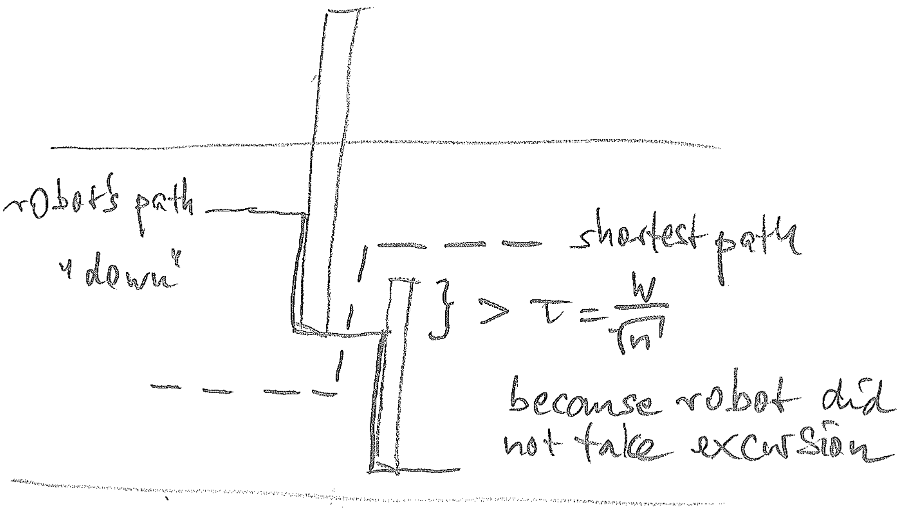
vertical motions:

if shortest path leaves strip



$\frac{W}{2}$

if shortest path stays within strip:
must cut through each of $\sqrt{n}$ "up's and "down"'s



$\sqrt{n} \cdot \frac{W}{\sqrt{n}}$

$n + \frac{W}{2}$

$\Rightarrow$ if no strip doubling occurs:

$$\frac{\text{robot's path}}{\text{shortest path}} \leq \frac{n + 3\sqrt{n}\,W}{n + \frac{W}{2}} \leq 6\sqrt{n}.$$

Now, suppose last strip is of size $\underbrace{2^j W_0}_{=: W_f}$, W_0 = size of first strip.

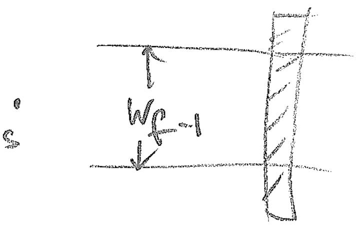
$$\underline{\text{robot's path}} \underset{\text{as before}}{\leq} n + \sum_{i=0}^{j} 2^i W_0 3\sqrt{n} < n + 2^{j+1} W_0 3\sqrt{n}$$
$$= n + 2 W_f 3\sqrt{n}$$

<u>shortest path:</u> horizontal: n

case 1: in previous strip of width $W_{f-1} = \frac{W_f}{2}$ all $\sqrt{n}$ up/down movements completed:

$$\underset{\text{as before}}{\Longrightarrow} \text{ shortest path} \geq \frac{W_{f-1}}{2} = \frac{W_f}{4}$$

case 2: W_f obtained, because large obstacle blocked strip of width W_{f-1}



$$\Rightarrow \text{ shortest path} \geq \frac{W_{f-1}}{2} = \frac{W_f}{4}$$

$$\Rightarrow \text{ shortest path} \geq n + \frac{W_f}{4}$$

$$\Rightarrow \frac{\text{robot's path}}{\text{shortest path}} \leq \underset{\text{caused by } L_1}{\sqrt{2}} \frac{n + 6 W_f \sqrt{n}}{n + \frac{W_f}{4}} \leq 36\sqrt{n}$$

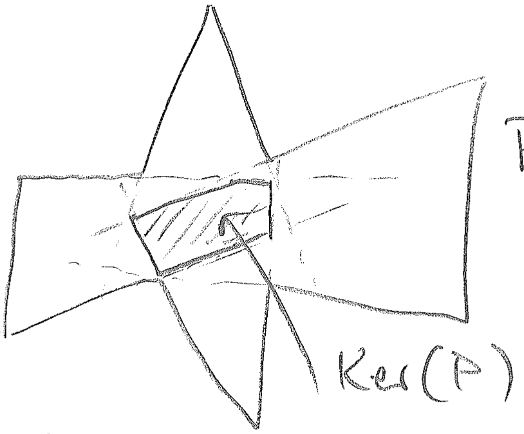
$\boxed{\text{Theorem}}$

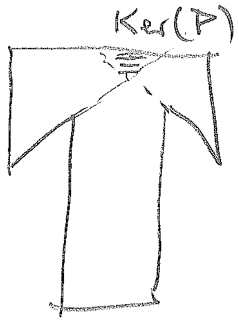
By previous theorem, this strategy is asymptotically optimal.

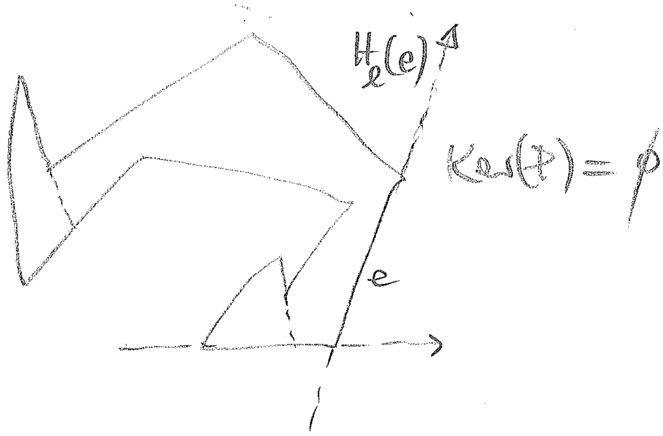
Back to visibility in simple polygons!

P: simple polygon of n edges

$$Ker(P) := \{ z \in P \mid \forall y \in P : z \text{ sees } y \}$$
$$= \{ z \in P \mid vis(z) = P \} \quad \text{Kernel of } P$$

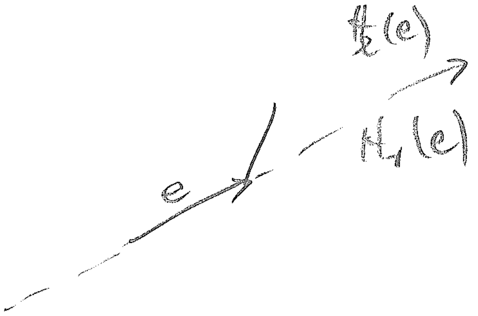






Let ∂P be counterclockwise oriented;

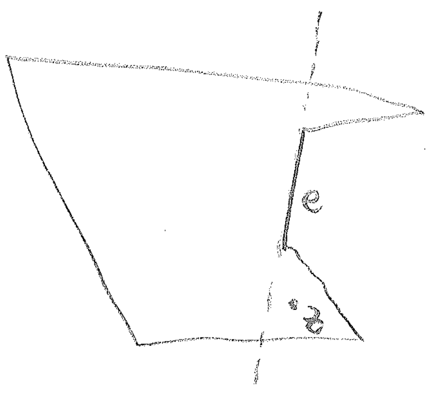
for each edge e:



$H_\ell(e)$:= half plane to the left of ray through e (closed)

<u>Lemma</u> $Ker(P) = \bigcap_{e \text{ edge of } P} H_\ell(e)$

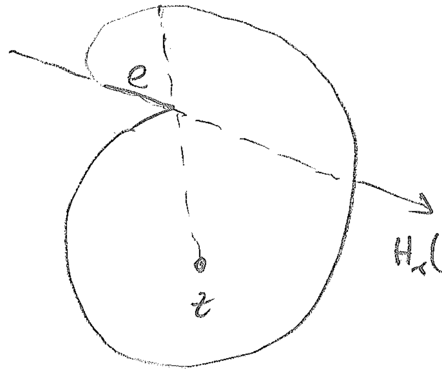
<u>Proof</u> „⊆" let $z \in Ker(P)$. If $z \in H_r(e)$ then z cannot see e:



$$\Rightarrow z \in \bigcap_e H_\ell(e)$$

"$\supseteq$" let $z \in \bigcap_e H_\ell(e) \Rightarrow z \in P$
(otherwise: z sees some edge e from the outside of P
$\Rightarrow z \in H_r(e)$ ↯)

Suppose $vis(z) \subsetneq P \Rightarrow$ there exists a cave of $vis(z)$:



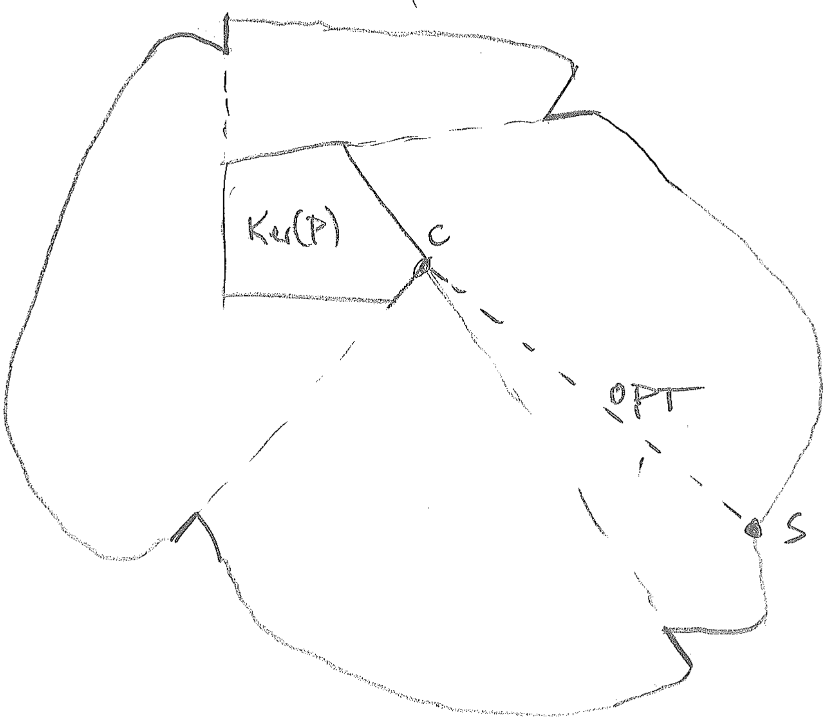
$\Rightarrow$ there exists edge e such that $z \in H_r(e)$ ↯.

$\boxed{\text{Lemma}}$

Def: P is called star-shaped $:\Leftrightarrow Ker(P) \neq \emptyset$.

Classic problem: Given P, determine if P is star-shaped. If so, compute $Ker(P)$
Can be solved in optimal time $O(n)$.

Here: On-line Problem: Given an unknown star-shaped polygon P, a start point $s \in \partial P$
move to closest point of $Ker(P)$ on as short a path as possible



Claim: (i) startpoint s can see Ker(P), but does not know, where it is

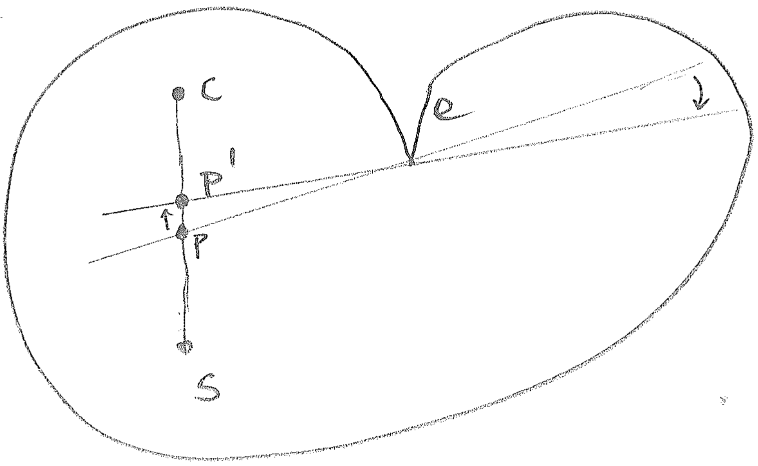
(ii) OPT is always a straight line segment (from s to nearest point of Ker(P))

(iii) As one moves along line segment OPT, vis(p) monotonically grows from vis(s) to P.

(i), (ii) are clear by definition. (iii)? Suppose visibility decreases at some point p of segment OPT:

⇒ c cannot see edge e

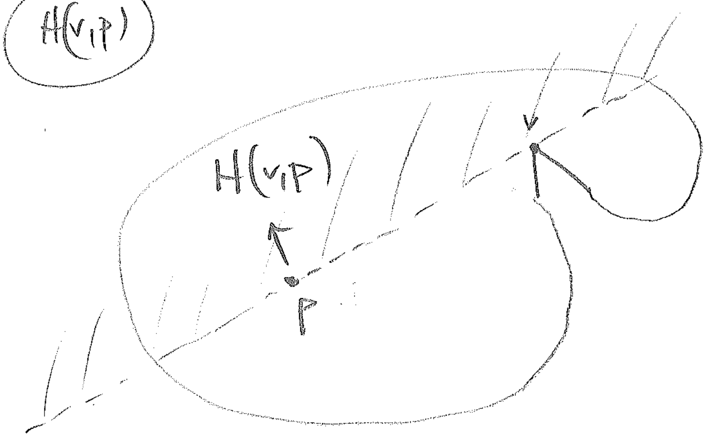
⇒ ∤ c ∈ Ker(P)



Idea: Let robot move in such a way that vis(p) keeps growing. How?

Let p = current position. Each cave of vis(p) defines a halfplane, into which we must move, in order to make vis(p) grow

H(v,p)



v = reflex vertex causing cave

Let $G(p) :=$ wedge at p, as defined above

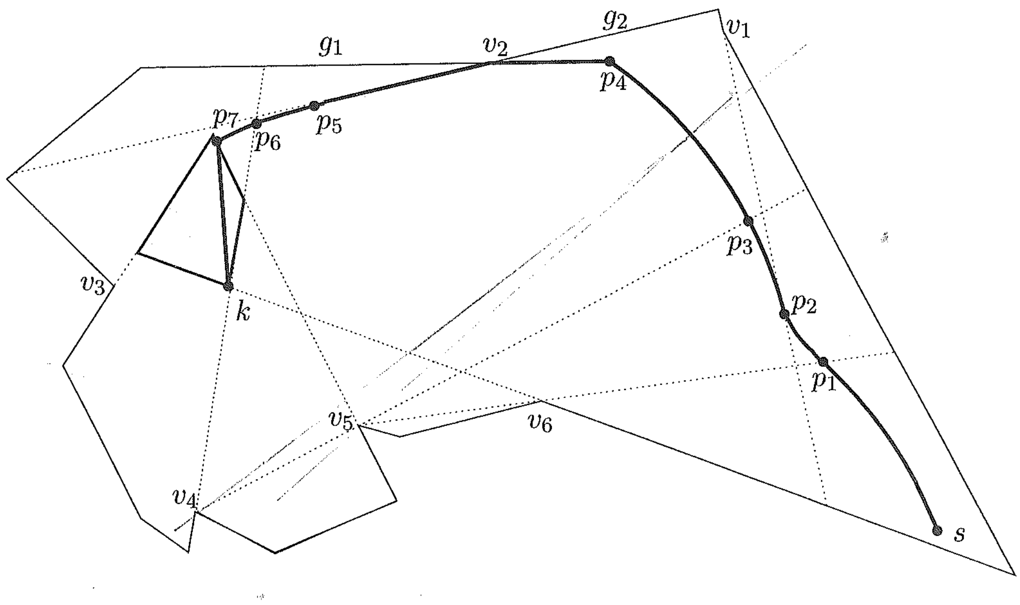
$E(p) := \bigcap_{p \in line(e)} H^l(e)$ ($= \mathbb{R}^2$ for all points p not contained in an edge extension)

Algorithm CAB

$p := s$;
repeat
    compute $G(p)$;
    $W :=$ angular bisector of $G(p)$;
    compute $E(p)$;
    if $W \subset E(p)$
        then follow W
        else follow projection of W onto boundary of $E(p$

until $p \in Ker(P)$;
walk straight to point $k \in Ker(P)$ closest to s.

Example :



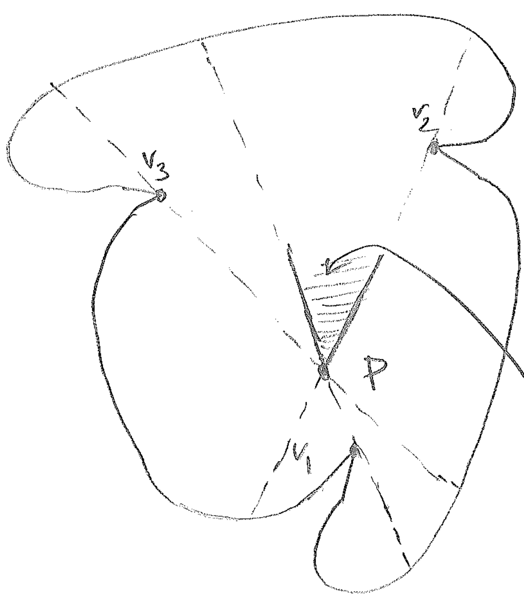
$s - p_1$: v_6, v_1 define $W(p)$ $\longrightarrow$ hyperbola

$p_1 - p_2$: v_5, v_1 " $\longrightarrow$ hyperbola

$p_2 - p_3$: only v_5 defines $W(p)$ $\longrightarrow$ circular arc

$p_3 - p_4$: v_4, v_5 define $W(p)$ $\rightarrow$ ellipse

$p_4 - p_5$: robot slides along boundary of $E(p)$ $\rightarrow$ strai[ght] segments before Ker(P) is reached, there may be several such halfplanes ;



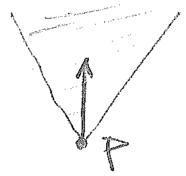
these intersection … a wedge, defined b… two halfplanes, inte… which robot must h…

$H(v_1,P) \cap H(v_2,P) \subset H(v_3,P$

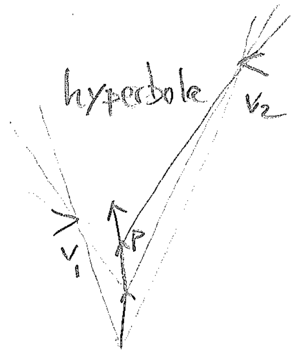
Continuos Angular Bisector

Strategy CAB : follow angular bisector into the wedge

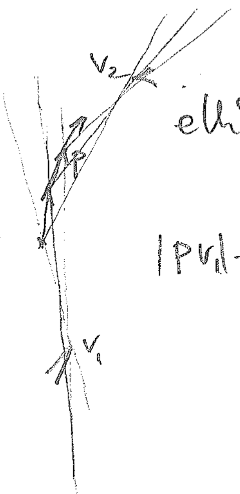
Clear: half planes defining wedge may change, as robot proceeds, as in street problem



As long as defining halfpl… do not change, the following… curves result :

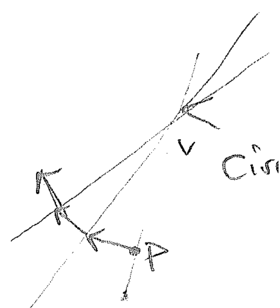


$|pv_2| - |pv_1| = \text{const}$



ellipse

$|pv_1| + |pv_2| = \text{Const}$

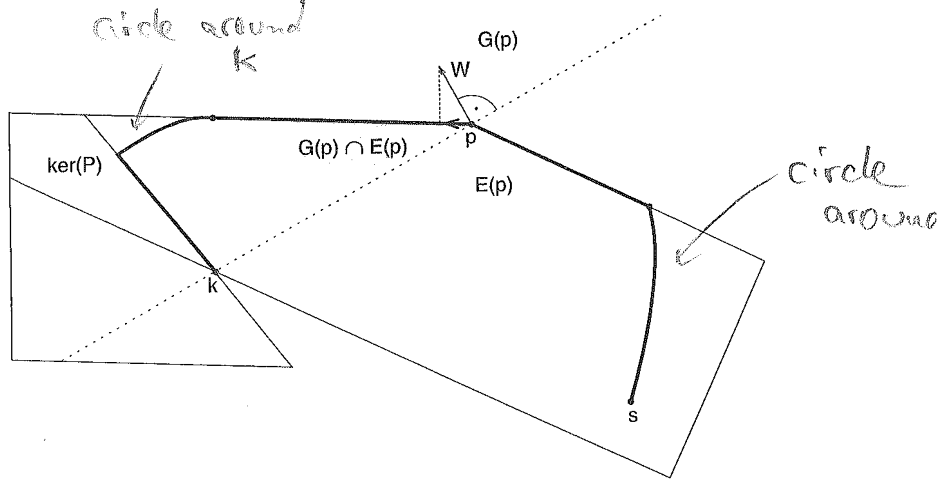


Cir…

$|pv| = \text{Cons}…$

While robot follows angular bisector it may hit a wall or an edge extension

circle around K



circle around…

$P_5 - P_6$: v_4, v_5 define $W(p)$ $\longrightarrow$ ellipse

$P_6 - P_7$: only v_5 defines $W(p)$ $\longrightarrow$ circular arc

$P_7 - k$: straight segment through $Ker(P)$.

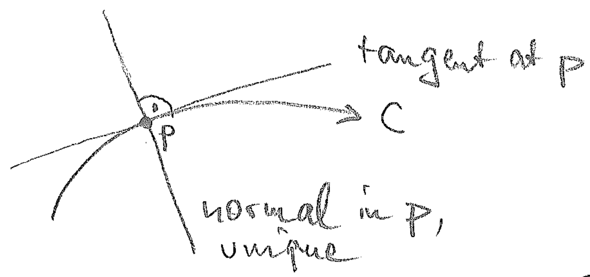
<u>How to analyze CAB paths ?</u>

\+ Only $O(n)$ segments $\leftarrow$ (each vertex discovered / explored at most once ; each edge extension visited only once while same vertices define $W(p)$)

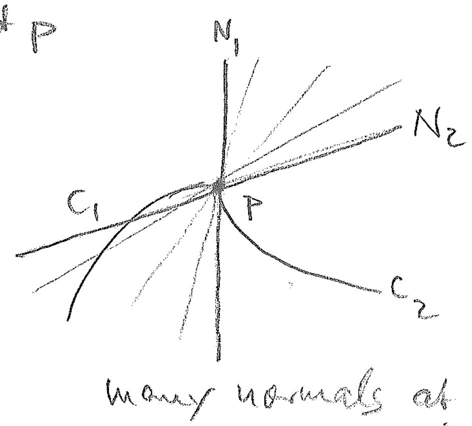
— no closed formula for length of elliptical arcs

But there is a nice structural property :

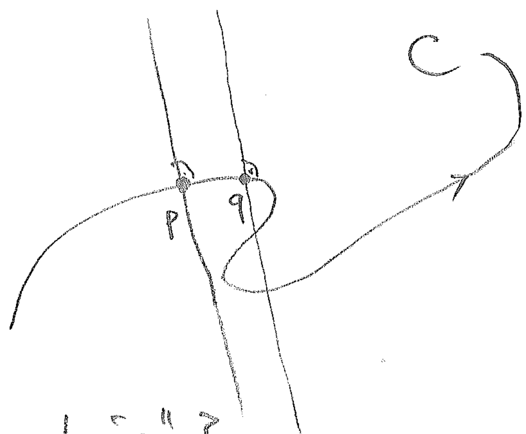
Let C be a smooth curve, $p \in C$:



Where two smooth curves meet:



<u>Def</u>: An oriented piecewise smooth curve C is called self-approaching
$:\Leftrightarrow$ $\forall p \in C$: rest of C lies in front of normal in p



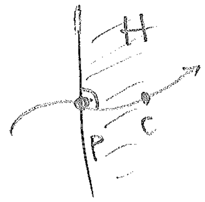
condition fulfilled for p, but not for q

$\Rightarrow$ C not self-approaching

Why "self-approaching" ?

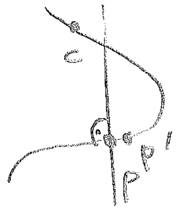
<u>Lemma 1</u> C self-approaching $\Leftrightarrow$ $\forall\, a, b, c \in C$ in this order :
$|bc| \leq |ac|$

Proof „$\Rightarrow$"



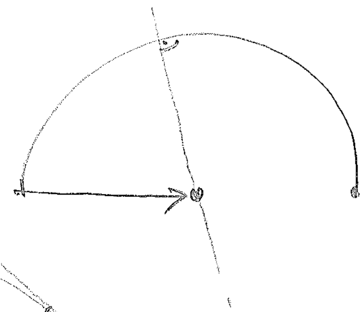
$\forall p$ before c : $|pc|$ is decreasing as p moves into half plane H

„$\Leftarrow$"

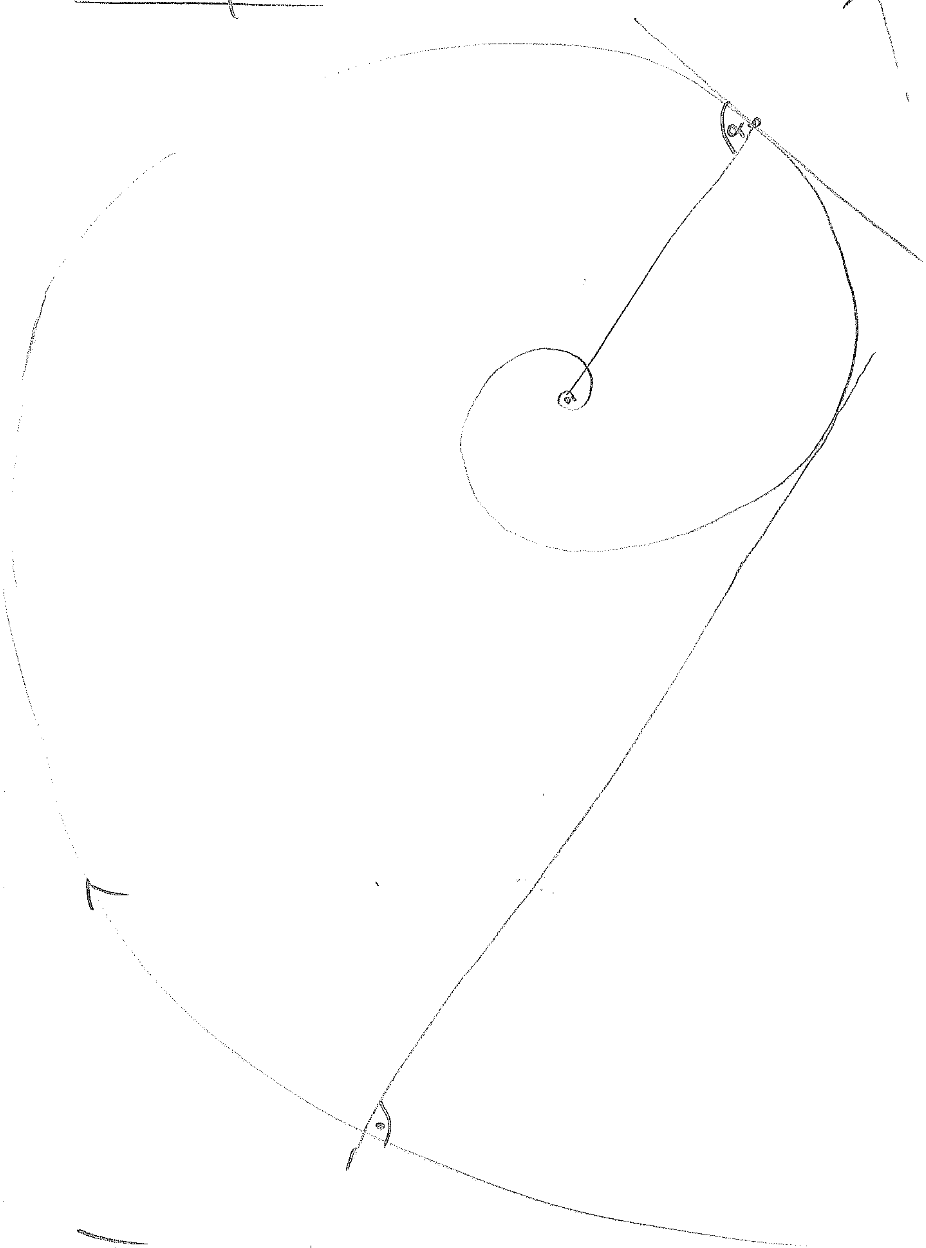


here, $|p'c| > |pc|$ ↯ $\boxed{\text{Lemma 2}}$

Examples



is self-approaching



$\alpha =$ const.
$= 74{,}66°$

exponential spiral
for this particular α:
each normal is a tangent

self-approaching

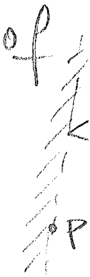
Theorem Each path created by strategy CAB is self-approaching.

Proof: Needs some auxiliary results.

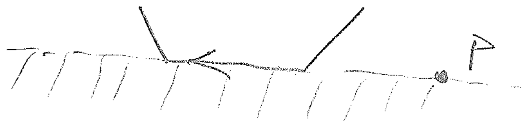
Lemma 2 In a sufficiently small neighborhood U of any point $p \in P$,

$$U \cap Ker(vis(p)) = G(p) \cap E(p) \cap U.$$

Proof By definition, $G(p) \cap E(p)$ is the intersection of



- left halfplanes defined by reflex vertices of $vis(p)$
- left halfplanes defined by edge extensions of P containing p

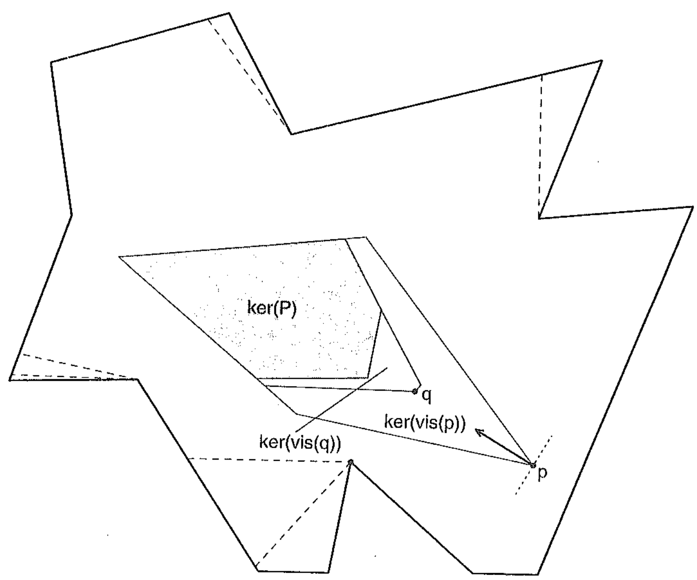


all these halfplanes are defined by edges of $vis(p)$.
By lemma 1, $Ker(vis(p))$ is the intersection of all left halfplanes of edges of $vis(p)$; but the remaining ones contain p in their interiors.

$\boxed{\text{Lemma 2}}$

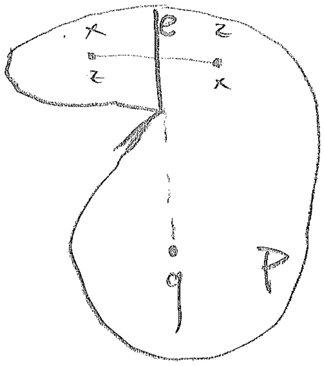
Lemma 3 Let $p, q \in P$ such that $vis(p) \subseteq vis(q)$.
Then

$$Ker(P) \underset{(i)}{\subseteq} Ker(vis(q)) \underset{(ii)}{\subseteq} Ker(vis(p)).$$



Proof (i) Let $z \in Ker(P)$ $\Rightarrow$ z sees q
$\Rightarrow$ $z \in vis(q)$

Let $x \in vis(q)$; $z \in Ker(P) \Rightarrow \overline{zx}$ not intersected by ∂
assume that an edge of $vis(q) \setminus P$ intersects $\overline{zx}$



$\Rightarrow$ q does not see x $\lightning_{x \in}$
or
q does not see z $\lightning_{z \in K}$

$\Rightarrow \overline{zx} \subset vis(q) \underset{\substack{x \in vis(q) \\ \text{arbitrary}}}{\Rightarrow} z \in Ker(vis$ $\boxed{(i)}$

(ii) Let $P' := vis_P(q)$. By assumption, $p \in vis_P(p) \subseteq vis_P(q)$

$\underset{(i)}{\Rightarrow}$ $Ker(vis_P(q)) = Ker(P') \subseteq Ker(vis_{P'}(p))$ (*)

clear: $vis_{P'}(p) \subseteq vis_P(p)$ since $P' \subset P$ ($\overline{zx} \subset P'$ $\Rightarrow \overline{zx} \subset P$)

the inverse is also true:

$x \in vis_P(p) \Rightarrow \overline{px} \subset vis_P(p) \underset{\uparrow \text{by assumption}}{\subseteq} vis_P(q) = P'$

$\Rightarrow x \in vis_{P'}(p)$

$\Rightarrow vis_{P'}(p) = vis_P(p)$

$\underset{(*)}{\Rightarrow}$ $Ker(vis_P(q)) \subseteq Ker(vis_P(p))$ $\boxed{(ii)}$ $\boxed{\text{Lemma 3}}$

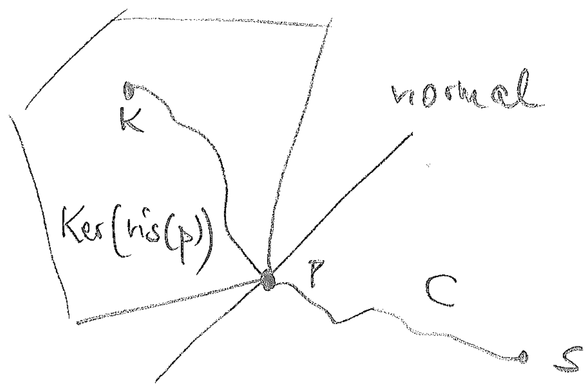
Corollary For each point p on a path generated by CA (s-k) (C)

$$C_p^K \subset Ker(vis(p)).$$

Proof By Lemma 3, $Ker(vis(p))$ keeps shrinking, as p moves along C_p^K. $\boxed{\text{Corollary}}$

Since $Ker(vis(p))$ lies "in front" of the normal in p, the theorem is proven: C is self-approaching.

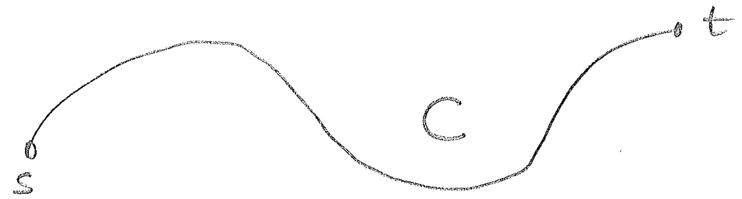
$\boxed{\text{Theorem}}$



Next step towards CAB - analysis: general result on self-approaching curves.

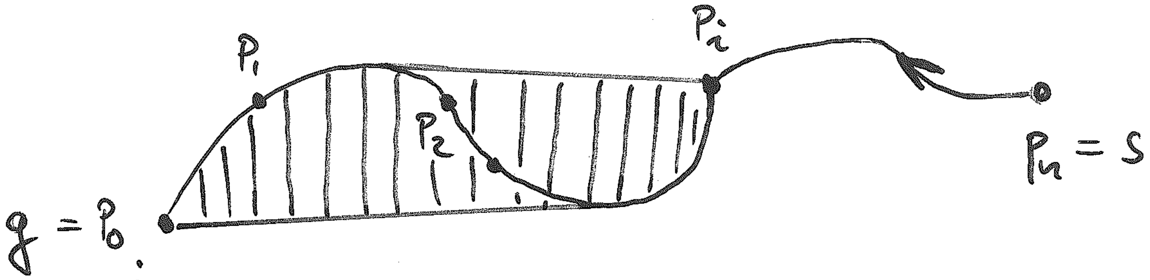
Theorem Let C be an oriented self-approaching curve from s to t. Then

$$\frac{|C|}{|st|} \leq 5.3331\ldots,$$ and this bound is tight.



Main Lemma $|C| \leq |\partial(ch(C))|$

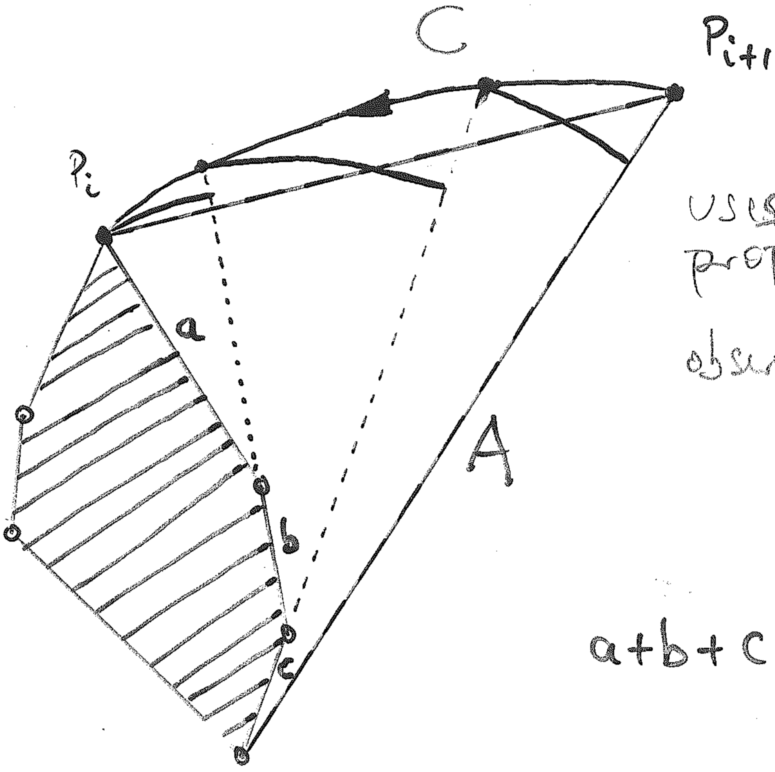
Proof (Sketch) By induction on i



$$\sum_{j=1}^{i} |p_j - p_{j-1}| \leq \text{perimeter}(\text{convhull}(p_0, \ldots, p_i))$$

case 1

p_i vertex of
ch($p_0, \dots, p_{i+1}$)



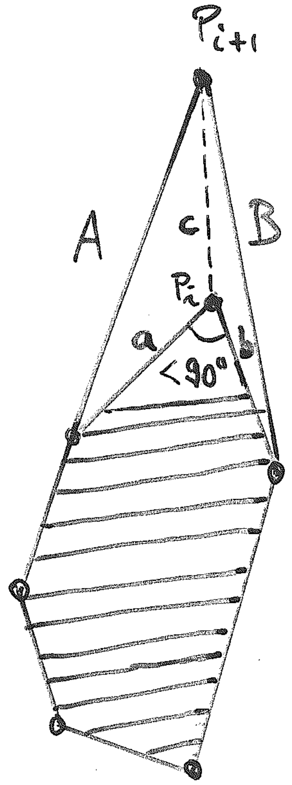
uses self-approaching property

observe: $\overline{p_i p_{i+1}}$ contributes to convex hull

$a+b+c \leq A$

case 2

$p_i \in$ ch($p_0, \dots, p_{i+1}$)



$a+b+c \leq A+B$