Proof:

We apply the randomized Min-Algorithm on

$w(P_1), w(P_2), \dots, w(P_r)$

Let $T(P)$ denote the running time for computing $w(P)$ (random variable)

$$N(P_i) = \begin{cases} 1, & w(P_i) \text{ has to be computed} \\ 0, & \text{otherwise} \end{cases} \qquad (0\text{-}1 \text{ variable})$$

We have: $E\left(\sum_{i=1}^{r} N(P_i)\right) \leq \ln r + 1$

as before choose uniformly at random (backwards) P_i's (same chance to be the minimum)

Now set:

$$\hat{t}(n) := \max_{|P| \leq n} E\left(T(|P|)\right)$$

maximal expected running time w.r.t. Problem size $\leq n$.

$P \in \Pi$, $|P| \leq n$

$$T(P) = \sum_{i=1}^{r} N(P_i)\, T(P_i) + O(r\, D(|P|))$$

r Tests

$N(P_i)$, $T(P_i)$ independent

Computation of $T(P_i)$ independent from the necessity of computation

$$\Rightarrow \quad \mathbb{E}(T(P))$$

$$= \sum_{i=1}^{r} \mathbb{E}(N(P_i))\,\mathbb{E}(T(P_i)) + O(r\,D(|P|))$$

$$\leq (\ln r + 1)\,\hat{T}(\alpha |P|) + O(D(|P|))$$

$\uparrow$ r is a constant!

Holds $\forall P \in \Pi$ with $|P| \leq n$

therefore

$$\hat{T}(n) \leq (\ln r + 1)\,\hat{T}(\alpha n) + R \cdot D(n) \quad \text{with constant } R$$

Show $\hat{T}(n) \leq C \cdot D(n)$ constant C

2. Cases: $(\ln r + 1)\alpha^{\varepsilon} < 1$ Case I.

By induction $\hat{T}(n) \leq C \cdot D(n)$ holds

$$\hat{T}(n) \leq (\ln r + 1)\,\underbrace{\hat{T}(\alpha n)}_{\leq C\,D(\alpha n) \text{ by Ind. Hypothesis}} + R \cdot D(n)$$

$$\left[\left(\frac{D(\alpha n)}{(\alpha n)^{\varepsilon}} \underset{\text{Monotinicity}}{\leq} \frac{D(n)}{n^{\varepsilon}} \right) \Rightarrow D(\alpha n) \leq \alpha^{\varepsilon} D(n) \right]$$

$$\leq (\ln r + 1)\alpha^{\varepsilon} \cdot C \cdot D(n) + R \cdot D(n)$$

$$\leq C \cdot D(n) \quad \text{for} \quad \frac{R}{\underbrace{1 - (\ln r + 1)\alpha^{\varepsilon}}_{> 0}} \leq C$$

Case II: $(\ln r + 1)\,\alpha^{\varepsilon} \geq 1$

Do recursion on P_i's ℓ-times!

$$r \longrightarrow r^{\ell} \qquad \alpha \longrightarrow \alpha^{\ell}$$

grows shrinks

$$\lim_{\ell \to \infty} \ln(r^{\ell}+1)\,\alpha^{\ell\varepsilon} = 0 \qquad \left(\alpha < 1 \text{ second term shrinks faster!}\right)$$

$$\Rightarrow \quad \ln(r^{\ell}+1)\,\alpha^{\ell\varepsilon} < 1 \quad \text{for some } \ell$$

$(r^{\ell}$ is the new $r)$ $\square$

Application to Dilation-Problem

Decision: $\delta(C) \overset{?}{\leq} t$ in $D(n) = n \log n$

$\dfrac{n \log n}{n^{\varepsilon}}$ grows

Generalization for decomposition into subproblems

U Set of Vertices of C

Q Set of Edges of C

Compute $\delta(U,Q) := \sup\Big\{ \delta(p,q) \;\Big|\; p \in U,\ q \in Q$ no segment of Q cuts pq $\Big\}$

Decision problem: $\delta(U,Q) \overset{?}{\leq} t$
in time $O(n \log n)$

Decomposition

$U = U_1 \dot\cup U_2$ Subsets of roughly the same size
$Q = Q_1 \dot\cup Q_2$

$\delta(U,Q) = \max \{ \delta(U_1, Q_1), \delta(U_2, Q_2), \delta(U_2, Q_1), \delta(U_1, Q_2) \}$

4 Subproblems of size $\frac{1}{2}(|U| + |Q|)$
$r = 4 \quad \alpha = \frac{1}{2}$!

Apply Chans technique!

Vertex / Vertex Problem Graph-theoretic Dilation

Use the same approach!

AVD Trace the chain and evaluation at the vertices!

Decision problem in $O(n \log n)$

Theorem 5 The (graph-theoretic and geometric) dilation of a polygonal chain can be computed in expected time $O(n \log n)$.

Lower bounds on the computation time!

Geometric dilation: $\Omega(n)$ $\quad$ $\Omega(n \log n)$ still unknown

Graph theoretic dilation $\Omega(n \log n)$
(Algebraic decision tree model)

Lemma 6 The graph-theoretic dilation of a polygonal chain has computational time $\Omega(n \log n)$.

Proof: Reduction of element uniqueness!

Element uniqueness $y_1, \ldots, y_n \in \mathbb{R}$

Does $i \neq j$ exists with $y_i = y_j$?

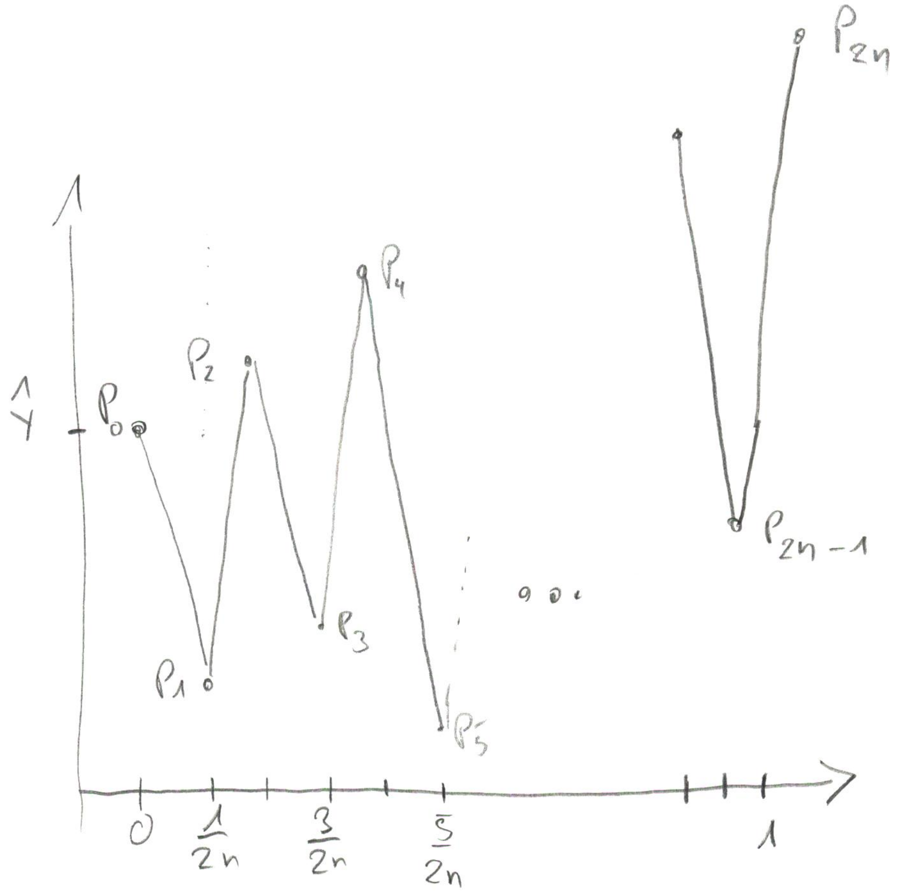
$\Omega(n \log n)$ $\quad$ Yao 1989

Compute $\delta_{graph}(C)$ faster than $\Omega(n \log n)$

$\Rightarrow$ answer Element uniqueness faster than $\Omega(n \log n)$

Reduction of $y_1, \ldots, y_n \rightarrow$ chain C

in $O(n)$!



$$P_{2i} = \left(\frac{2i}{2n}, \hat{Y} + i\right) \qquad i = 1, \dots, n$$

$$P_{2i-1} = \left(\frac{2i-1}{2n}, Y_i\right) \qquad i = 1, \dots, n-1$$

$$\overline{Y} := \max_{1 \le i \le n} Y_i \qquad \underline{Y} = \min_{1 \le i \le n} Y_i$$

Idea: Maximum dilation for points

If $Y_i = Y_j \Rightarrow$ max dilation between points P_{2i-1} and P_{2j-1}

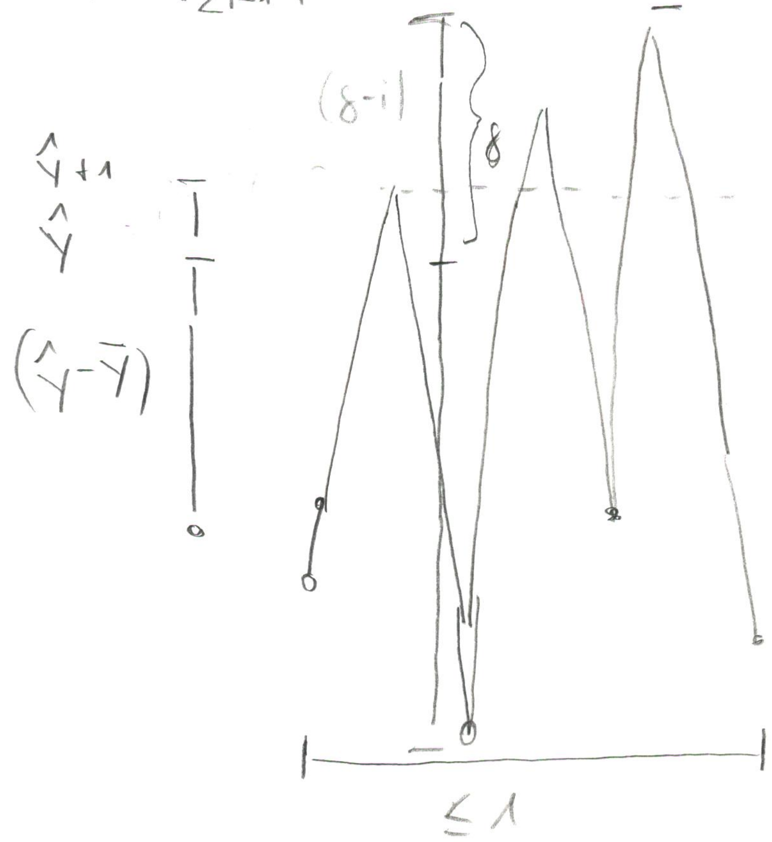
$j > i$:

$$\left|P_{2j-1}P_{2i-1}\right| = \sqrt{\left(\frac{2j-1}{2n} - \frac{2i-1}{2n}\right)^2 + (y_j - y_i)^2}$$

$$\begin{cases} = \dfrac{2(j-i)}{2n} & \text{if } y_j = y_i \\ > \sqrt{\left(\dfrac{2(j-i)}{2n}\right)^2 + 1} & \text{if } y_j \neq y_i \end{cases}$$

$$\left|\Pi_{P_{2i-1}}^{P_{2j-1}}\right| > 2(\hat{y} - \bar{y})(j-i) + 2(j-i) \qquad \text{I.}$$

$$\left|\Pi_{P_{2i-1}}^{P_{2j-1}}\right| < 2(\hat{y} - \underline{y} + \delta + \tfrac{1}{2})(j-i) \qquad \text{II.}$$



I. $2(\hat{y} - \bar{y} + 1)$

$(j-i)$ times

II. $2(\hat{y} - \underline{y} + \delta)$ $+ (j-i)$

$(j-i)$ times

Choose $\hat{y}$ so that:

$$\frac{2\left(\hat{y}-\underline{y}+j+\frac{1}{2}\right)(j-i)}{\sqrt{\left(\frac{2(j-i)}{2n}\right)^2+1}} < \frac{2\left(\hat{y}-\bar{y}+1\right)(j-i)}{\frac{2(j-i)}{2n}}$$

$$\text{"}\Longleftarrow\text{"}$$

$$\frac{2\left(\hat{y}-\underline{y}+n+\frac{1}{2}\right)n}{\sqrt{\left(\frac{1}{n}\right)^2+1}} < 2\left(\hat{y}-\bar{y}+1\right)\cdot n$$

$$\Longleftrightarrow$$

$$\frac{\left(\hat{y}-\underline{y}+n+\frac{1}{2}\right)^2}{\left(\hat{y}-\bar{y}+1\right)^2} < \underbrace{\left(\frac{1}{n^2}\right)+1}_{=\,D(n)\ \text{fix}}$$

$\longrightarrow 1$

for $\hat{y} \longrightarrow \infty$

Dilation between

P_{2i} and P_{2j}

Similarly $\leq \dfrac{2\left(\hat{y} - \underline{y} + \delta + \frac{1}{2}\right)(j-i)}{\sqrt{\left(\frac{2(j-i)}{2n}\right)^2 + 1}}$

This means: Construct chain

$C(y_1, y_2, \ldots, y_n)$ with corresponding $\hat{y}$ in

$O(n)$ time.

Apply Alg for graph-theoretic dilation:

Pair (P_k, P_ℓ) attains maximum

If $P_k.y = P_\ell.y$ then element uniqueness

$y_1, \ldots, y_n$ "Yes"

otherwise "no"

$\square$

Some remarks:

1) Lower bound for geometric dilation of a chain $\Omega(n)$ $\Omega(n \log n)$?? Open problem

2) Deterministic approximation $(1+\varepsilon)$ approx

$O\left(\frac{1}{\varepsilon} n \log n\right)$ Grüne et al.

3) Dilation of cycles (closed chains), trees

$O(n \log^2 n)$ Difference:

Separation pair: Agarwal et al. 09

4) Dilation planar Graph with n vertices

$O\left(n^2 \left(\frac{\log \log n}{\log n}\right)^4\right)$ Wulff-Nilsen 09