Online Motion Planning MA-INF 1314
Restricted Graphexploration

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Rep: Restricted Graphexploration

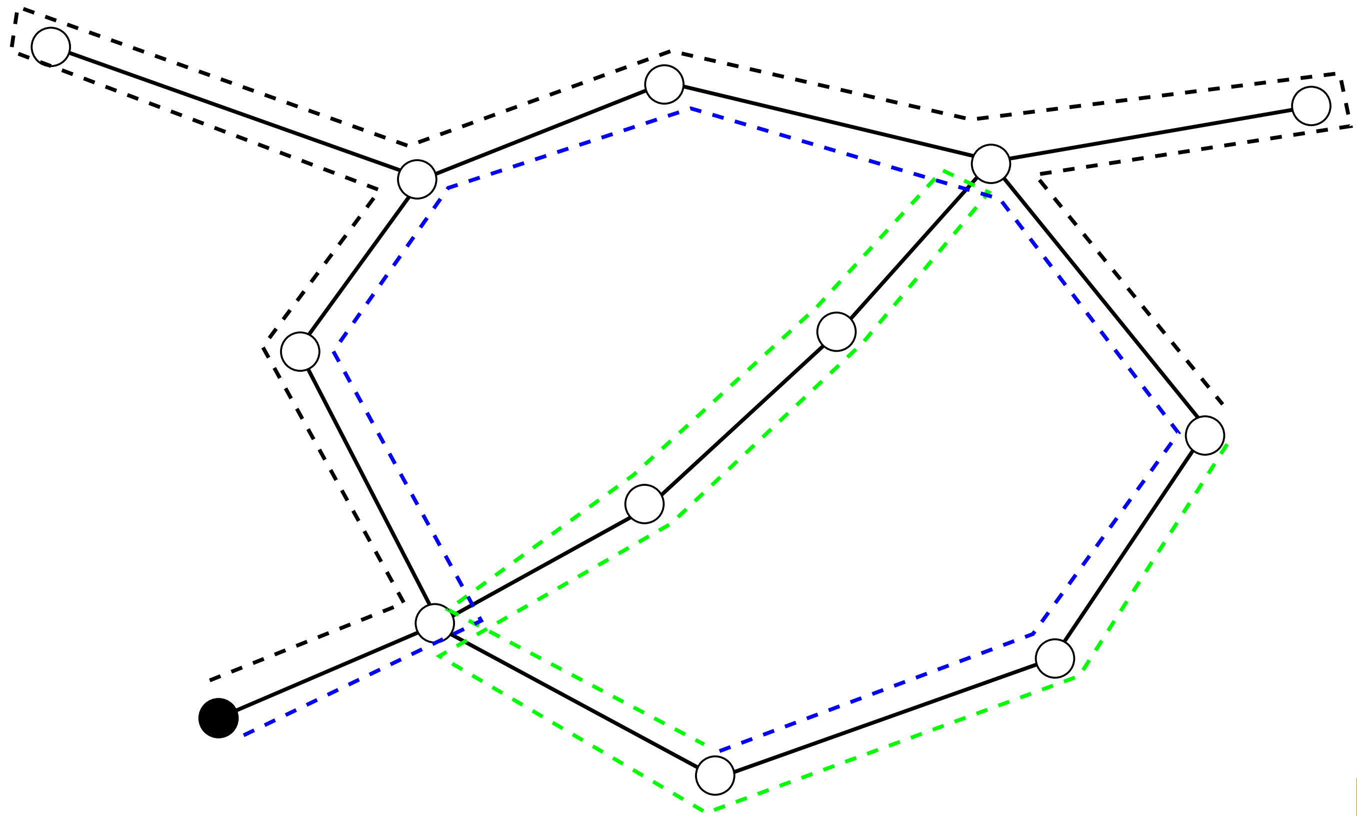
1. Agent with tether of length $l = (1 + \alpha)r$ (i.e., cable)
2. Agent has to return after $2(1 + \alpha)r$ steps (recharge accumulator)
3. Large graph is explorde up to bounded depth d.

- Graph has depth r
- Unit length edges
- Small α

Repetition

- Emulation: Tether variant given $l = (1+\alpha)r$, cost T
- **Lemma:** Accumulator-variant, $2(1+\beta)r$, cost $\frac{1+\beta}{\beta-\alpha}T$
- After $2(\beta-\alpha)r$ break and return
- Offline Problem, Accumulator-variant, NP-hard?
- **Lemma** $6|E|$ Offline-Approximation bei Accumulator size 4r
- Cut DFS in 2r parts, move to, work, return and so on

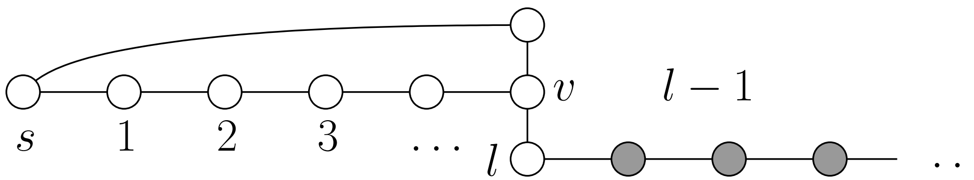
Repetition: Offline example/Refinement



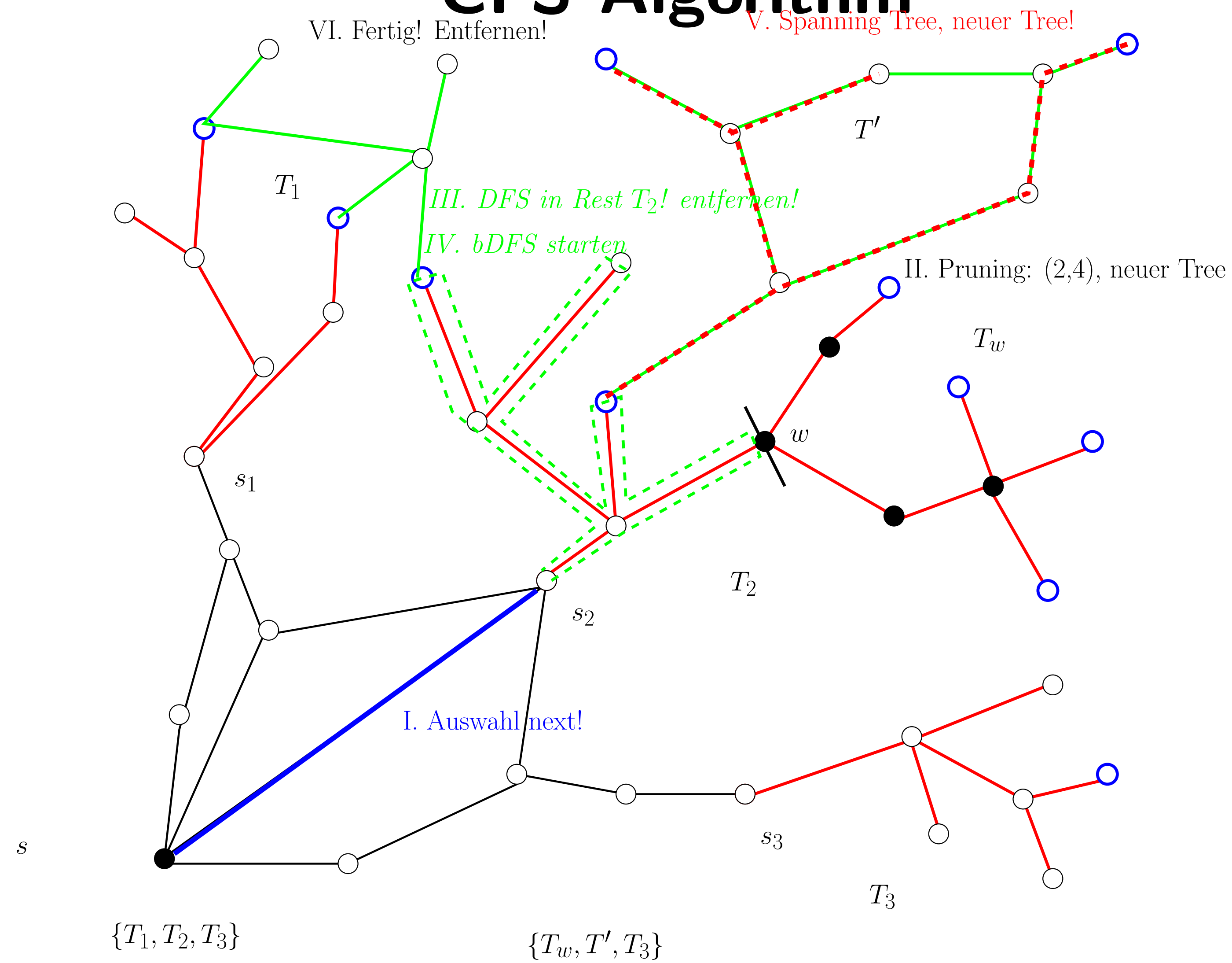
$$\left\lceil \frac{2|E|}{2r} \right\rceil \times 2r + 2|E| \leq 6|E|$$ Example: $r = 5$

Idea: Bounded DFS

- Example unit-length edges
- v fully exlored, tether has ended, backtracking
- Not only bDFS



CFS Algorithm



CFS Algorithm

- Start bDFS at different sources
- Set of (edge) disjoint **trees** $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$
- Root vertices $s_1, s_2, \ldots, s_k$
- Choose T_i with s_i closest to s, move to s_i
- Pruning of T_i: Build T_{w_j} with root w_j if:

1. $d_{T_i}(s_i, w_j) \geq \mathit{minDist} = \frac{\alpha r}{4}$
2. $\mathit{Depth}(T_{w_j}) \geq \mathit{minDepth} - \mathit{minDist} = \frac{\alpha r}{4}$

- Add all T_{w_j} to $\mathcal{T}$! Remove T_i from $\mathcal{T}$
- Explore T_i without T_{w_j} from s_i by DFS and
- start bDFS at the incomplete vertices

- Graph G' of new vertices and edges
- Build a spanning tree T' of G
- Choose root s' with minimal distance to s
- Add all these trees to $\mathcal{T}$
- Special case: Trees in $\mathcal{T}$ gets fully explored
- Trees in $\mathcal{T}$ with common egdes are joined
- Merging: Build spanning tree with new root

CFS Algorithm: Invariants Lemma

Execution CFS–Algorithm, properties hold:

i) Any incomplete vertex belongs to a tree in $\mathcal{T}$.

ii) There is always an incomplete vertex with $v \in V^*$ with $d_{G^*}(s, v) \leq r$, until $G^* \neq G$.

iii) For any chosen root vertex s_i: $d_{G^*}(s, s_i) \leq r$.

iv) After pruning T_i is fully explored by DFS. All trees $T \in \mathcal{T}$ have size $|T| \geq \frac{\alpha r}{4}$.

v) All trees $T \in \mathcal{T}$ are disjoint (w.r.t. edges)

Proof: i) and v) simply hold by construction

Proof invariant ii)

$G \neq G^*$: Ex. incomplete vertex with $v \in V^*$ with $d_{G^*}(s,v) \leq r$

- Ass: For all incomplete $v \in V^*$: $d_{G^*}(s,v) > r$
- Choose v incomplete with minimal $d_{G^*}(s,v)$
- Shortest path to v in G: $d_G(s,v) \leq r$
- SP is partially inside G^*, first incomplete v' of G^*
- Inside V^* and incomplete
- $d_{G^*}(s,v') = d_G(s,v') \leq r < d_{G^*}(s,v)$
- Contradiction to choice of v!!

Proof invariant iii)

For any chosen root vertex s_i: $d_{G^*}(s, s_i) \leq r$.

Any incomplete vertex of T_i *behind* s_i

Statement follows directly from (ii)!

Proof invariant iv)

After pruning T_i is fully explored by DFS.

By iii) distance $d_{G^*}(s, s_i) \leq r$

Visit vertex v of T_i

$$r(1+\alpha) - d_{G^*}(s, s_i) - d_{T_i}(s_i, v) \geq \frac{\alpha r}{2}$$

since $d_{T_i}(s_i, v) \leq \frac{\alpha r}{2}$ by pruning

Proof invariant iv)

All trees $T \in \mathcal{T}$ have size $|T| \geq \frac{\alpha r}{4}$.

- Start: bDFS from s, finished or $|T| \geq (1+\alpha)r > \frac{\alpha r}{4}$
- Successively: T_i, so that s_i root closest to s
- $|T_i| \geq \frac{\alpha r}{4}$
- T_i after pruning
- Pruning guarantees: $|T_i| \geq \frac{\alpha r}{4}$
- T_w cut-off by pruning: $|T_w| \geq \frac{\alpha r}{2} - \frac{\alpha r}{4} = \frac{\alpha r}{4}$
- New bDFS-Trees: $d_{G^*}(s, s_i) \leq r$, $d_{T_i}(s_i, v) \leq \frac{\alpha r}{2}$ (pruned)
- At least length $\frac{\alpha r}{2}$ of tether remains for bDFS-Trees
- New spanning tree $|T'| \geq \frac{\alpha r}{4}$
- All $T \in \mathcal{T}$ have been considered

Analysis Theorem

CFS–Algorithmus for restricted graph-exploration of an unknown graph with known depth r is $(4 + \frac{8}{\alpha})$–competitive.

Subtree T_R

- Subtree T_R, cost
- $K_1(T_R)$: path from s to s_i in G^*
- $K_2(T_R)$: Exploration by DFS
- $K_3(T_R)$: bDFS starting from incomplete vertices (Graph!)
- $\sum_{T_R} K_3(T_R) \leq 2 \cdot |E|$, bDFS only visits unexplored edges
- $\sum_{T_R} K_2(T_R) = \sum_{T_R} 2 \cdot |T_R| \leq 2 \cdot |E|$, the cost of DFS

Analysis Theorem

- Subtree T_R, cost
- $K_1(T_R)$: path from s to s_i in G^*
- $K_1(T_R) = 2 \cdot d_{G^*}(s, s_i) \leq 2r$, **Lemma**
- Pruning: Komplexity T_R at least $\frac{\alpha r}{4}$
- $|T_R| \geq \frac{\alpha r}{4}$, which gives $r \leq \frac{4|T_R|}{\alpha}$, **Lemma**
- $\sum_{T_R} K_1(T_R) \leq \sum_{T_R} 2r \leq \frac{8}{\alpha} \sum_{T_R} |T_R| \leq \frac{8}{\alpha}|E|$
- $2 \cdot |E| + 2 \cdot |E| + \frac{8}{\alpha}|E|$ versus $|E|$
- $\leq (4 + \frac{8}{\alpha})|E|$

Corollary

CFS–Algorithm for restricted graph-exploration of an unknown graph of known depth requires $\Theta(|E| + |V|/\alpha)$ steps.

- $K_3(T_R)$: bDFS starting from incomplete vertices, visit edges
- $K_1(T_R)$ and $K_2(T_R)$ analysed by TREES T_R
- vertices and edges same size, vertex could appear in 2 trees!
- $\left(2 + \frac{8}{\alpha}\right) 2|V|$ and $2|E|$ only for bDFS

Unknown graph of unknown depth!

- Assumption: Depth R is not known
- Heuristik: Double r successively
- So that finally r is large enough!
- Start with $r := 2$, $\approx \log_2 R$ calls
- Means $O(\log R|E|)$ steps
- Re-exploration by bDFS could be avoided, trees!
- $O(|E| + (\log R)|V|)$ steps, **Corollary**

Unknown graph of unknown depth R!

- Improvement!
- Adjusting pruning and explore steps
- Depth and tether by circumstances: $d_{G^*}(s, s_i)$ is current r
- Algorithm still has cost: $\left(2 + \frac{8}{\alpha}\right) 2|V| + 2|E|$
- prune(T_i, s_i, $\frac{\alpha r}{4}$, $\frac{\alpha r}{2}$)
- Substitution by: prune(T_i, s_i, $\frac{\alpha d_{G^*}(s,s_i)}{4}$, $\frac{9\alpha d_{G^*}(s,s_i)}{16}$)
- explore($\mathcal{T}$, T_i, s_i, $(1+\alpha)r$)
- Substitution by: explore($\mathcal{T}$, T_i, s_i, $(1+\alpha)d_{G^*}(s, s_i)$)

$d_{G^*}(s, s_i)$ substitute of r for explore/prune

- Start-problem: $d_{G^*}(s, s_i) = 0$, $r := \max(d_{G^*}(s, s_i), c)$
- Structural properties **Lemma**

  i) Any incomplete vertex belongs to a tree in $\mathcal{T}$.

  ii) There is always an incomplete vertex with $v \in V^*$ with $d_{G^*}(s, v) \leq r$, until $G^* \neq G$.

  iii) For any chosen root vertex s_i: $d_{G^*}(s, s_i) \leq r$.

  iv) After pruning T_i is fully explored by DFS. All trees $T \in \mathcal{T}$ have size $|T| \geq \frac{\alpha r}{4}$.

  v) All trees $T \in \mathcal{T}$ are disjoint (w.r.t. edges)

- i), ii), iii) and v) also true for adjusted calls

Properties for adjusted explore/prune

$r := \max(d_{G^*}(s, s_i), c)$ where $d_{G^*}(s, s_i)$ minimal for $T_i \in \mathcal{T}$

Lemma:

i) Any incomplete vertex belongs to a tree in $\mathcal{T}$.

ii) There is always an incomplete vertex with $v \in V^*$ with $d_{G^*}(s, v) \leq r$, until $G^* \neq G$.

iii) For any chosen root vertex s_i: $d_{G^*}(s, s_i) \leq r$.

iv) After pruning T_i is fully explored by DFS. All trees $T \in \mathcal{T}$ have size $|T| \geq \frac{\max(d_{G^*}(s,T),c)\alpha}{4}$.

v) All trees $T \in \mathcal{T}$ are disjoint (w.r.t. edges)

iv) has to be shown!

Proof invariant iv)

After pruning: T_i is fully explored by DFS

Distance $d_{G^*}(s, s_i)$

Visit vertex v of T_i

$$(1 + \alpha)d_{G^*}(s, s_i) - d_{G^*}(s, s_i) - d_{T_i}(s_i, v) \geq \frac{7d_{G^*}(s, s_i)\alpha}{16}$$

since $d_{T_i}(s_i, v) \leq \frac{9d_{G^*}(s,s_i)\alpha}{16}$ by pruning!

Proof invariant iv)

$\forall T \in \mathcal{T} : |T| \geq \frac{\max(d_{G^*}(s,T),c)\alpha}{4}$

- Start: bDFS from start, finished or $|T| \geq (1+\alpha)c > \frac{\alpha c}{4}$
- Otherwise T_i, with s_i root closest to s, Ass.: $d_{G^*}(s, T_i) > c$!
- Show: $|T_w| \geq \frac{d_{G^*}(s,T_w)\alpha}{4}$
- Tree T_i after pruning, $d_{G^*}(s, T_i) = d_{G^*}(s, s_i)$
- Pruning guarantees: $|T_i| \geq \frac{d_{G^*}(s,T_i)\alpha}{4}$
- T_w cut-off by pruning: $|T_w| \geq \frac{9d_{G^*}(s,s_i)\alpha}{16} - \frac{d_{G^*}(s,s_i)\alpha}{4} = 5\frac{d_{G^*}(s,s_i)\alpha}{16}$
- $d_{G^*}(s, T_w) \leq d_{G^*}(s, s_i) + d_{G^*}(s_i, w) = (1 + \frac{\alpha}{4})d_{G^*}(s, s_i) < \frac{5d_{G^*}(s,s_i)}{4}$, $\alpha < 1$!
- Gives: $|T_w| > \frac{d_{G^*}(s,T_w)\alpha}{4}$

Proof invariant iv)

$\forall T \in \mathcal{T} : |T| \geq \frac{\max(d_{G*}(s,T),c)\alpha}{4}$

- Spanning trees T' after bDFS from v in T_i
- bDFS-trees after v: $d_{G^*}(s_i, v) \leq \frac{9\alpha d_{G*}(s,s_i)}{16}$
- $d_{G^*}(s, T') \leq d_{G^*}(s, s_i) + d_{G^*}(s_i, v) < \frac{25 d_{G*}(s,s_i)}{16}$
- $d_{G^*}(s, s_i) > \frac{16 d_{G*}(s,T')}{25}$
- T' incomplete
- $d_{G^*}(s_i, v) \leq \frac{9\alpha d_{G*}(s,s_i)}{16}$
- At least $\frac{7\alpha d_{G*}(s,s_i)}{16}$ tether rest
- $|T'| \geq \frac{7\alpha d_{G*}(s,s_i)}{16} > \frac{7\alpha d_{G*}(s,T')}{25} > \frac{d_{G*}(s,T')\alpha}{4}$
- Either explored or $|T'| > \frac{d_{G*}(s,T')\alpha}{4}$
- Any $T \in \mathcal{T}$ was considered

Analysis Theorem

CFS–Algorithmus for restricted graph-exploration of unknown graph with unknown depth is $(4 + \frac{8}{\alpha})$–competitive for $0 < \alpha < 1$.

Trees T_R

- Tree T_R, cost
- $K_1(T_R)$: path from s to s_i in G^*
- $K_2(T_R)$: Exploration by DFS
- $K_3(T_R)$: bDFS from incomplete vertex (Graph!)
- $\sum_{T_R} K_3(T_R) \leq 2 \cdot |E|$, since bDFS visits only unexplored edges
- $\sum_{T_R} K_2(T_R) = \sum_{T_R} 2 \cdot |T_R| \leq 2 \cdot |E|$, cost for DFS

Analyse Theorem 1.30

- Teilbaum T_R, Kosten
- $K_1(T_R)$: path from s to s_i in G^*
- $K_1(T_R) = 2 \cdot d_{G^*}(s, s_i) \leq \frac{8|T_R|}{\alpha}$
- $|T_R| \geq \frac{d_{G^*}(s,T_R)\alpha}{4} = \frac{d_{G^*}(s,s_i)\alpha}{4}$, **Lemma iv)**
- $\sum_{T_R} K_1(T_R) \leq \sum_{T_R} 2d_{G^*}(s, s_i) \leq \frac{8}{\alpha} \sum_{T_R} |T_R| \leq \frac{8}{\alpha}|E|$
- $2 \cdot |E| + 2 \cdot |E| + \frac{8}{\alpha}|E|$ against E

Corollary

CFS–Algorithmus for restricted graph-exploration of unknown graph with unknown depth performs $\Theta(|E| + |V|/\alpha)$ steps for $0 < \alpha < 1$.