Problem Set 11

Problem 1

Let E be a set of m elements (for example, think of edges in a graph $G = (V, E)$) and let $\mathcal{S}$ be a family of subsets of E (for example, think of all feasible matchings in G). Suppose that each $e \in E$ has a weight $w(e)$, and this weight is chosen uniformly at random from $\{1, \ldots, L\}$. Let the weight of a set (a matching) $M \in \mathcal{S}$ be defined as $w(M) = \sum_{e \in M} w(e)$.

Let M^* be an element from $\mathcal{S}$ with maximum weight (a maximum matching), i.e. $w(M) \leq M^*$ for all $M \in \mathcal{S}$. Prove that the probability that M^* is the unique element with weight $w(M^*)$ is at least $1 - \frac{m}{L}$. In other words, prove

$$\Pr(\exists M' \in \mathcal{S} \backslash M^* : w(M') = w(M^*)) < \frac{m}{L}.$$

Problem 2

To prove Theorem 8.1 from the lecture, show that in any round i of the SSP algorithm, the flow f_i is a minimum-cost flow among all flows with value $|f_i|$.

To do so, use the following theorem: A flow f is of minimum cost among all flows of value $|f|$ if and only if G_f contains no cycle with negative total weight.

Problem 3

Discuss how the SSP algorithm performs on the following input network. For each edge, the first number is the capacity, the second number is the cost. The value T is a given integer.