Optimal online escape path against a certificate[*] [**]

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Abstract. More than fifty years ago, Bellman asked for the best escape path within a known forest but for an unknown starting position. This deterministic finite path is the shortest path that leads out of a given environment from any starting point. There are some worst case positions where the full path length is required. Up to now such a fixed *ultimate optimal escape path* for a known shape for any starting position is only known for some special convex shapes (i.e., circles, strips of a given width, fat convex bodies, some isosceles triangles).

Therefore, we introduce a different, simple and intuitive escape path, the so-called *certificate path*. This escape path depends on the starting position s and takes the distances from s to the outer boundary of the environment into account. Due to the additional information, the certificate path *always* (for any position s) leaves the environment earlier than the ultimate escape path, in the above convex examples.

Next we assume that fewer information is available. Neither the precise shape of the environment, nor the location of the starting point is known. For a class of environments (convex shapes and shapes with kernel positions), we design an *online* strategy that always leaves the environment. We show that the path length for leaving the environment is always shorter than 3.318764 the length of the corresponding certificate path. We also give a lower bound of 3.313126, which shows that for the above class of environments the factor 3.318764 is (almost) tight.

1 Introduction

We consider the following motion planning task. Let us assume that we are given a simple polygon P and a starting point s inside P. We would like to design a simple path starting at s that finally hits the boundary and leaves the polygon. In the sense of a game, we can choose a path but then an adversary can rotate the polygon P around s so that the path will leave the polygon very late.

First, we assume that the distance from s to the boundary is given into every direction. We can apply a simple and intuitive strategy. The *certificate path* is the best combination of a line segment l and an arc of length $l\alpha$ along

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the circle of radius l around the starting point. So this path simply checks an angular portion of the environment for a distance l. For a given starting point the certificate path is the best (shortest) such path that guarantees to hit the boundary. Altogether the certificate path is a very simple *escape path* for given s and P (if an adversary can only rotate P around s).

In turn, for any given unknown starting position s inside an unknown polygon, we would like to design an *online strategy* (based on fewer information) that is never much worse than the length of the above certificate path. In this paper, we show that for a class of environments, there is a spiral strategy that leaves any such polygon and approximates the length of the certificate path within a ratio of 3.318674. We also prove that this is an (almost) tight bound. There is no other strategy that always attains a better ratio against the length of the certificate path.

This optimal online approximation is restricted to the following class of environments. We assume that in any direction from the unknown starting point only one boundary point exists. The distance to the boundary points still remains unknown. This subsumes any unknown convex environment (for any unknown starting position) and also unknown star-shaped environments (for any unknown starting point inside the kernel). The motivation of comparing an online escape path for special polygons (star-shaped) and special starting positions (inside the kernel) with a path that is computed with some additional but not complete information (certificate path) stems from the following observation.

For a known polygonal shape and an unknown starting point, it is possible to define an *ultimate optimal escape path*. This path will lead out of the environment for any starting point and any rotation of the polygon. The ultimate optimal escape path is the shortest finite path with this property. The clue is that only the polygon is known but neither the starting position, nor the rotation around the starting position. The path is motivated by the situation of swimming in the fog in a pool of known shape. As it is foggy, the starting point and the rotation around the starting point is not known. Unfortunately, ultimate optimal escape paths have been found only for a few special convex shapes (circles, strips of given width, fat convex bodies, isosceles triangles, ...). It is unrealistic to think that such paths will be found for more complicated convex or star-shaped environments.

Fortunately, for the few cases where an ultimate optimal escape path is known, the certificate path is not only a good approximation. We can even show that the certificate path *beats* the ultimate escape paths for *any* starting point in these examples. Therefore, we are convinced that the certificate path can serve as a substitute for the unknown ultimate optimal escape path in the restricted cases.

The paper is organized as follows. In the next section we present related work. The certificate path is introduced and defined in Section 3. Different justification for the measure is discussed in Section 3.2. Finally, in Section 4, we present and analyse a strategy with path length not larger than 3.318674 times the length of the certificate path. The strategy is a logarithmic spiral attained by keeping

aware of two extremes of the certificate. Optimizing the spiral for two extremes is also different from classical logarithmic spiral constructions where we normally optimize against a single distance (shortest path). In Section 5, we present a general lower bound which proves that the given strategy is almost optimal for the restricted cases. No other strategy will have a better ratio than 3.313126 against the length of the certificate path. Proving lower bounds is a tedious task, the construction and the analysis might be interesting in its own right.

2 Related work

The *Swimming-in-the-fog* problem is a game where two players, a searcher and a hider, compete with each other. The searcher tries to reach the boundary of a known shape from its starting point along a single finite path, while the hider can rotate and translate the environment so that the path of the searcher will cross the boundary as late as possible. For a given shape, the shortest finite path that always leads out of the given environment can be denoted as an *ultimate optimal escape path* as mentioned before.

Since the first work by Koopman in 1946, search games have been studied in many variations in the last 60 years. The book by Gal [14] and the reissue by Alpern and Gal [1] gives a comprehensive overview of such search game problems, also for unknown environments.

The above problem goes at least back to 1956 and to Bellman [3], who similarly asked for the shortest escape path within a known forest but for an unknown starting point. Since then, the problem has attracted a lot of attention. Unfortunately until today, the problem could be solved only for very special convex environments (circles, strips of given width, rectangles, fat convex bodies, isosceles triangles); see for example the monograph of Finch and Wetzel [10].

For circles and fat convex bodies, it was shown that the diameter is the ultimate optimal escape path; see Finch and Wetzel [10]. For the infinite strip of width l, the ultimate optimal escape path is due to Zalgaller [26,27]. For the simple equilateral triangle of side length 1, the zig-zag path of Besicovitch [4] of length ≈ 0.981981 is optimal; see also [6]. Furthermore, in 1961 Gluss [15] introduced the problem of searching for a circle C of given radius s and given distance r away from the start A. Two different cases can be considered, either A is inside C or not. Interestingly, in the latter case and for $s = 1$ a certificate path with length $l = r$ and an arc of length $2\pi \cdot l$ is the best one can do.

It is unrealistic to think that such ultimate optimal escape paths will be found for more complicated environments. As an alternative, we introduced a simple and natural *certificate path* for a known environment and a given starting point s. It is computed individually for any starting point and takes the distance distribution from s to the boundary into account. Fortunately, for the cases where ultimate optimal escape paths are known, they are always outperformed by the certificate path for any possible starting point. This means we can prove that the certificate path always leaves the environment earlier.

The use of alternative comparison measures has some tradition. For example for the problem of searching for a point in a polygon and competing against the shortest path, there is no competitive strategy. For this reason other comparison measures have been suggested in this case; see Fleischer et al. [12] or Koutsoupias et al. [19]. Additionally, comparing the online strategy to the shortest path to the boundary is often a very difficult task. For example, the spiral conjecture for searching for a single line or a single ray against the shortest path is still open. In this sense our result might be considered as an intermediate step.

For the design of the online escape path, we assume that neither the precise shape of the environment, nor the position of the starting point is known. We make use of the competitive framework to show that the online strategy can compete with the certificate path, which is computed with more information. That is, we compare the length of the online escape path from a starting point to the boundary to the length of the certificate path to the boundary computed for the known environment and starting point. The competitive framework was introduced by Sleator and Tarjan [25], and used in many settings since then; see for example the survey by Fiat and Woeginger [8] or, for the field of online robot motion planning, see the surveys [16,23].

Our optimal online approximation is restricted to the following class of environments. We assume that in any direction from the unknown starting point, only one boundary point exists. The distance to the boundary points remains unknown. This subsumes any unknown convex environment (for any unknown starting position) and also unknown star-shaped environments (for any unknown starting point inside the kernel). In this sense, the certificate is also a natural extension of the discrete performance measure Kirkpatrick [17] mentioned in the discrete case of searching for the end of a set of m given lists of unknown length. In his setting it is sufficient to reach the end of only one list. In our configuration, this means that we have exactly 2π directions of unknown distance and it is sufficient to reach the shoreline in a single point. The corresponding relationship is shown in Section 3.2.

We will see that our solution is a specific logarithmic spiral. In general, logarithmic spirals are natural candidates for optimal competitive search strategies, but in almost all cases the optimality remains a conjecture; see [2,7,9,11,14]. In [20] the optimality of spiral search was shown for searching a point in the plane with a radar. Many other conjectures are still open. For example Finch and Zhu [11] considered the problem of searching for a line in the plane. *The relevant conjecture that the family of logarithmic spirals contains the minimal path remains open.*

3 The certificate path

Assume that you are located in an unknown environment and would like to reach its boundary. Formally, for the environment we consider a closed Jordan curve B that subdivides the Euclidean plane into exactly two regions. The starting
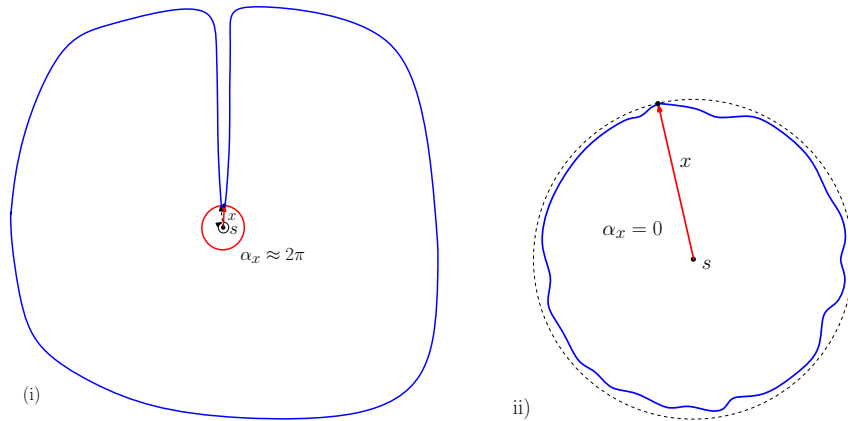
Fig. 1. Two extreme situation for reaching the boundary with a circular arc. (i) The radial maximal distance from s to the boundary is almost the same in any direction. It suffices to move in an arbitrary direction of maximal distance, which is optimal. (ii) The distance to some few boundary points is very small, while large to most of the others. A reasonable path checks the small distance with a circular arc of length approximately 2π. In both cases, $x(1 + \alpha_x)$ is minimal among all such circular strategies.

point s lies inside the inner region, say P. The task is to reach a point on the boundary B, as soon as possible.

If you have some idea about the distance x from s to the boundary B but nothing more, it is very intuitive to move along the circle of radius x around the starting point. Therefore, a reasonable strategy moves toward this circle along a shortest path (by radius x) in some direction and then follows the circle in either clockwise or counterclockwise direction until the boundary is met. Let us denote this behaviour a *circular strategy*. If we hit the boundary after moving an arc α_x along the circle, the overall path length is given by $x(1 + \alpha_x)$.

We would like to use such a circular strategy of small path length. In the sense of a game, the adversary can only rotate the environment around the starting point and the certificate path guarantees to hit the boundary for any rotation.

3.1 Extreme cases and general definition

Let us first consider two somehow extreme examples of the above intuitive idea as given in Figure 1. If the distance from s to the boundary is almost the same in any direction (similar to a circle), a line segment with maximal distance to the boundary (roughly the radius of the circle) will always hit the boundary and is indeed a very good escape path for any direction; see Figure 1(ii). The movement along an arc is not necessary in this case. In other words, α_x equals 0. We check a single direction for the largest distance.

On the other hand, it might be the case that the distance to the boundary is very large w.r.t. almost all, but small (distance x) for some few directions from

s. Then, a segment of length x and a circular arc of length $x\alpha_x$ with $\alpha_x \approx 2\pi$ will hit the boundary for any starting direction of the segment x; see Figure 1(i). The overall path length $x(1 + \alpha_x)$ is comparatively small. The certificate path checks a small distance for many (almost all) directions.

Now, consider a more general environment modelled by a simple polygon P and a fixed starting point s in P as given in Figure 2(ii). For convenience, we make use of an example, where any boundary point b of P is *visible* from s, i.e. the segment sb lies fully inside P. Or, the other way round, s lies inside the kernel of P.
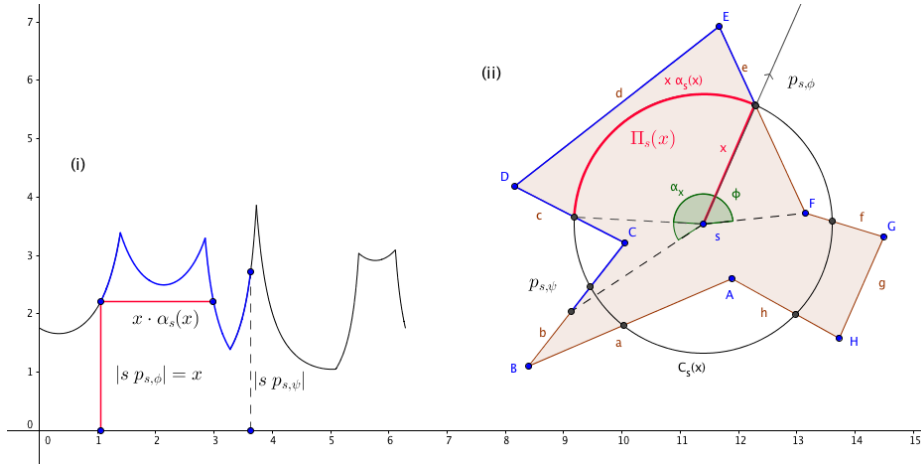

Fig. 2. (ii) Consider the polygon P and a starting point s. Let us assume that we radially sweep the boundary of P (starting from point F with angle 0) in counterclockwise order and calculate the distance from the boundary to s for any angle. (i) shows this *radial distance function* of the boundary of P from s in polar coordinates for the interval $[0, 2\pi]$. The blue sub-curve corresponds to the blue boundary part in (ii). The certificate path $\Pi_s(x)$ for distance x is the longest path that successfully checks the distance x by a circular strategy. This means that it hits the boundary for any starting direction ϕ of x in P. In the polar-coordinate setting in (i) this is a path with two line segments of length x and α_x that always hits the boundary of the radial distance function independent from the starting angle ϕ.

For the polygon P and for any radial direction $\phi \in [0, 2\pi]$ from s, we consider the boundary point $p_{s,\phi}$ on P in direction ϕ. This gives a *radial distance function* $f(\phi) := |s\, p_{s,\phi}|$, as depicted in Figure 2(i).

Now, let $p_{s,\phi}$ be a point with distance $x := |s\, p_{s,\phi}|$ in direction ϕ. For any circle $C_s(x)$ with radius x around s such that $C_s(x)$ hits the boundary of P, there will be some maximal arc $\alpha_s(x)$ so that the above simple circular strategy is successful. Note, that this is independent from the starting direction for x. We are looking for the maximum circle segment of $C_s(x)$ that fully lies inside P.

Let $\Pi_s(x)$ denote the *certificate path for distance x* of maximal length $x(1 + \alpha_s(x))$. The interpretation is that this finite path will always touch the boundary, independent from the starting direction for x. The adversary can only rotate the environment in order to attain a worst case length of $x(1 + \alpha_x)$.

Every certificate path for a distance x corresponds to two connected segments in the plot of the radial distance function ; see Figure 2(i). The vertical segment of length x represents the radius and the horizontal segment of length α_x represents the arc of the circular strategy. For any starting angle, this path will touch the boundary of the distance function.

We define he overall *certificate path Π_s* in P for a given starting point s as the shortest certificate path $\Pi_s(x)$ among all distances x. That is, the certificate for P and s is:

$$\Pi_s := \min_x \Pi_s(x) = \min_x x(1 + \alpha_s(x)) \ .$$

For both extreme situations in Figure 1, the presented paths equal the overall certificate paths for the given environments.

Finally, consider the case when parts of the boundary are not visible from s. In this case the radial maximal distance computation is no longer a function, it results in a curve; see Figure 3. However, the certificate $\Pi_s(x) = x(1 + \alpha_x)$ for distance x, the maximal arc α_x and the overall certificate Π_s are still well defined. Note that, for the certificate Π_s in any polygon P and the corresponding arc α_x, we can conclude $\alpha_x \in [0, 2\pi]$. This holds since the shortest distance d_s from s to the boundary always results in a candidate $d_s(1 + 2\pi)$. All other reasonable distances x are larger than d_s and $\alpha_x \leq 2\pi$ holds for the optimal x.

The online approximation of the certificate path of an arbitrary unknown polygon with a spiral strategy cannot be competitive in general. The corresponding ratio could grow arbitrarily large, as the polygon might wind itself around the spiral. In more general environments other online strategies have to be applied. For example, one could think of a connected sequence of circles C_i with exponentially increasing radii r^i. This would give at least a constant competitive ratio. However, obtaining the optimal strategy for such cases might be hopeless.

3.2 Justification of the certificate

The certificate path is an intuitive and simple way of leaving an environment. It can be computed in polynomial time, as we show in A.1. We can interpret the certificate as a path that balances depth-first and breadth-first search for the starting position s in a way that the resulting path is as short as possible. That way, it outperforms the ultimate optimal escape path at any given starting position for all known cases.

For circles, semi-circles with an opening angle α larger than 60 degrees and for *fat* convex bodies, the ultimate optimal escape path equals the diameter; see Figure 4. For any position s the worst case for this escape path is given by a rotation of the environment so that a segment of maximal length x is required.
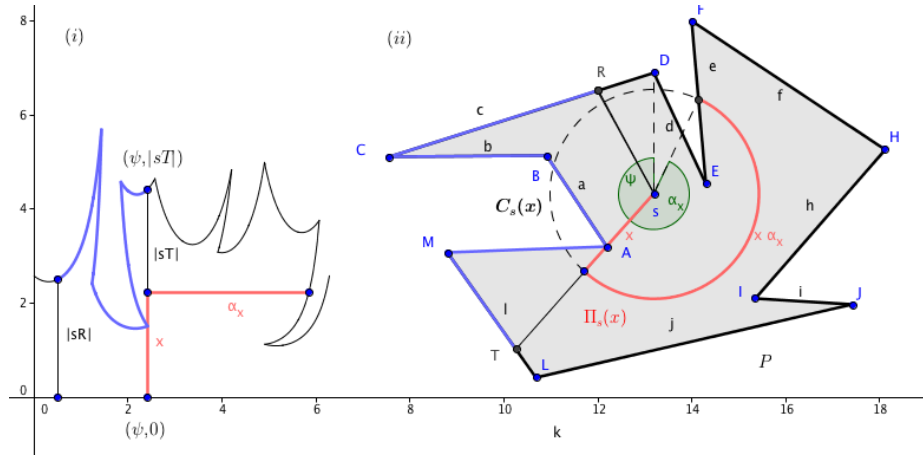
Fig. 3. (ii) For a general polygon P and a point $s \in P$ not the whole boundary might be visible. (i) The radial distance computation appears to be a curve. Nevertheless, the certificate for distance x and also the overall certificate is well-defined and has the same geometric interpretation.

As the certificate path Π_s considers such a path as a possible alternative, the certificate path is as least as good as the diameter for any position.

For the case when the environment is an equilateral triangle or an infinite strip, the certificate outperforms the ultimate optimal escape path for any starting position in the worst case. As this is not straight forward to see, we give small proofs in the following.

The Equilateral Triangle & Besicovitch's path. Consider an equilateral triangle as depicted in Figure 5. Besicovitch's zig-zag path is the ultimate optimal escape path for this environment; c.f. the discussion by Besicovitch [4] and the proof of optimality by Coulton and Movshovich [6]. The zig-zag path is symmetric and consist of three segments of length $\sqrt{3/28}$ each. This constitutes a total length of ≈ 0.981918. An example of a worst case starting point X_1 is given in Figure 5.

Consider the following observation. For starting points somewhere in the center of the triangle, the zig-zag path is worse than the largest distance to the boundary; see for example starting point s_1 in Figure 5. Thus, the certificate is shorter for these points. For starting points close to the boundary the certificate is significantly better by a short circular check; see for example starting point s in Figure 5. Now, we give a formal proof, that the zig-zag path can never beat the certificate. The values for $\alpha = \arcsin\left(1/\sqrt{28}\right) \approx 10.9°$ and $x = \sqrt{3/28}$ are due to Coulton and Movshovich [6].

If Besicovitch's zig-zag path does not hit a point on the boundary with maximum distance away from the starting point, it does not hit one of the three vertices of the triangle. Only in this case Besicovitch's zig-zag path can beat the
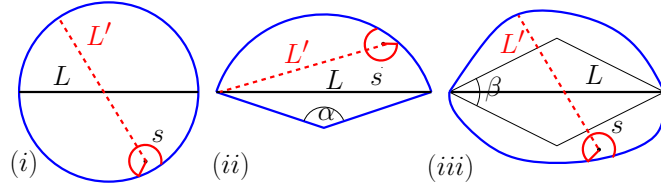
Fig. 4. Three environments, where the diameter L is the ultimate optimal escape path. (i) A circle, (ii) a semi-circle with angle $\alpha \geq 60°$. (iii) A so-called *fat* convex body. An environment is called fat if a rhombus with angle β of least $60°$ fits into it. Then, the diameter of the rhombus equals the diameter of the environment. The usage of the ultimate optimal escape path (line segment L) from s results in the largest distance x to the boundary (dashed path L') in the worst case. The certificate path is as least as good as the diameter path.

certificate. In this case, the Besicovitch path has to make use of at least two segments (each of length $x = \sqrt{3/28}$) for leaving the triangle. As shown in Figure 6, only a small area close to a vertex of the triangle has to be checked. For all those points, we find a circular strategy that is shorter than $2x$, which is always required by the zig-zag path. Note that we can easily extend this argumentation to the family of isosceles triangles analysed by Coulton and Movshovich [6,22].

The Infinite Strip & Zalgaller's path. Consider the infinite strip of width l. Zalgaller's path ζ is an ultimate optimal escape path; see [26], [27] and an alternative proof in [5]. It consists of four line segments and two arcs, which are defined by the following values; see Figure 7.

$$\phi = \arcsin\left(\tfrac{1}{6} + \tfrac{4}{3}\sin\left(\tfrac{1}{3}\arcsin\tfrac{17}{64}\right)\right)$$

$$\psi = \arctan\left(\tfrac{1}{2}\sec\phi\right)$$

$$x = \sec\phi \approx 1.043590$$

The total length from Z_1 to the end at Z_7 along ζ is approximately $2.278292\,l$. The path is symmetric with regard to the bisector of the baseline. The baseline itself has length $x \cdot l$.

To prove that the certificate path outperforms Zalgaller's escape path for any given starting point in the infinite strip, we proceed in two steps. At first, we show that for any starting position escaping along ζ takes at least $2.15 \cdot l$ in the worst case. Then, we show that, in the worst case, the length of the certificate path is always below this bound.

For the first argument, we argue that ζ can always be rotated around the starting point appropriately. Figure 7 shows the shape of ζ on the left-hand side. W.l.o.g. we assume that we follow ζ in clockwise orientation and name the vertices of ζ from Z_1 to Z_7 appropriately. We denote d the distance from the staring point s (which equals Z_1) to the lower bound of the strip. We may
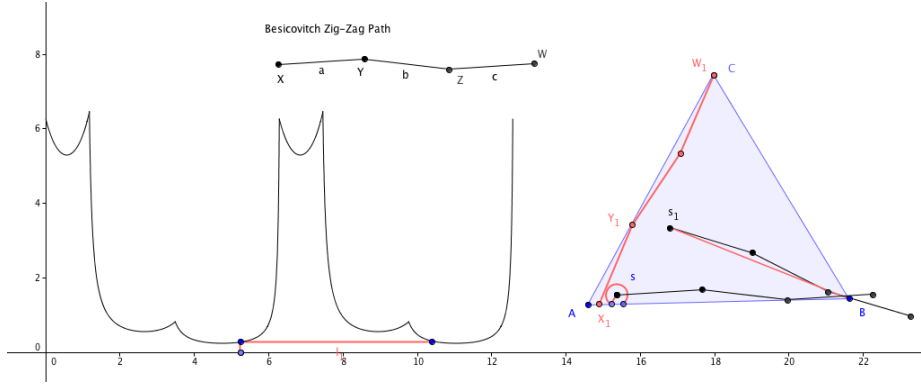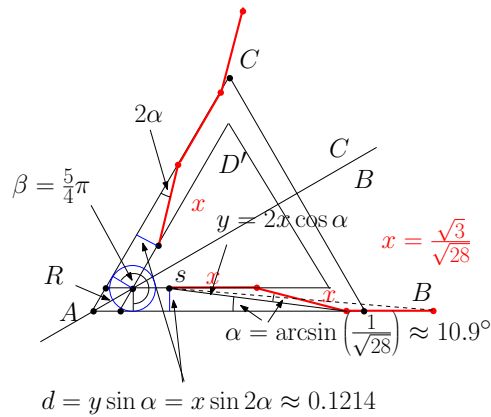
Fig. 5. Besicovitch's zig-zag path is the ultimate optimal escape path for an equilateral triangle. A worst case position is given by X_1 for example. The plot on the left-hand side shows the radial distance curve of point s. The certificate path, for such points close to the boundary, is very short. In contrast to this, the usage of Besicovitch's path is much worse, as the worst case is attained when leaving the triangle on the boundary farthest away.

assume that $d \in [0; l/2]$, otherwise we can turn the whole configuration by $180°$. Now, we rotate ζ around Z_1 so that Z_4 lies on the upper bound of the strip and Z_3, Z_5 both lie inside the strip; see Figure 7 on the right-hand side. This is always possible as the following argument shows. As $\triangle Z_3 Z_4 Z_5$ is equal-sided, the segment (Z_3, Z_5) is parallel to the baseline and $\angle Z_4 Z_5 Z_1 = 90°$, we have $\angle Z_5 Z_3 Z_4 = 90° - 2\phi$. The congruency of $\triangle Z_1 Z_5 Z_4$ and $\triangle Z_1 Z_4 X$ allows to conclude $\angle Z_3 Z_4 Z_1 = 2\phi - \psi$. On the one hand, this proves that Z_3 indeed lies in the strip as $\angle Z_3 Z_4 Z_1 = 2\phi - \psi < \arcsin\left(d \cdot (0.25x^2 + 1)^{-1/2}\right) = \angle U Z_4 Z_1$ for any $d \in [0; l/2]$. On the other hand, we can also show that Z_5 lies inside the strip as $\angle U Z_4 Z_5 < \angle U Z_4 Z_1 + \angle Z_3 Z_4 Z_5 = \arcsin\left(d \cdot (0.25x^2 + 1)^{-1/2}\right) + 4\phi < 180°$ for any $d \in [0; l/2]$.

Now that we know that this configuration can be always be attained, we consider point E, where ζ leaves the strip. Summing up the length of ζ from Z_1 to E exceeds the lower bound of 2.13, which gives the proof. Due to symmetry, the lower bound also holds if Z_7 lies in the infinite strip and we follow ζ counterclockwise.

For the second argument, we consider certificate paths for two different distances, which perform better than ζ for a given starting position. Again, we denote d the distance from the starting position to the lower bound of the strip and assume w.l.o.g. that $d \in [0; l/2]$. We split this interval in half. For starting positions with $d \in [0, l/4]$, we know that $\Pi_s(d)$ is a certificate path and α_d is a full circle. In this case, the length of Π_s is well below $2.11l$. Otherwise, we have $d \in [l/4, l/2]$. In this case, we consider the certificate path for distance l. The length of $\Pi_s(l)$ is strictly decreasing with growing $d \in [l/4, l/2]$. Consequently,

$$\beta = \tfrac{5}{4}\pi$$

$$2\alpha$$

$$x$$

$$y = 2x\cos\alpha$$

$$x = \frac{\sqrt{3}}{\sqrt{28}}$$

$$\alpha = \arcsin\left(\frac{1}{\sqrt{28}}\right) \approx 10.9°$$

$$d = y\sin\alpha = x\sin 2\alpha \approx 0.1214$$

$$0.125 \times (5\pi/4 + 1) < 2x = 2\frac{\sqrt{3}}{\sqrt{28}}$$

Fig. 6. For comparing the certificate and the Besicovitch's zig-zag path we only have to consider the case when the triangle cannot be rotated so that the zig-zag path ends in the vertex farthest away. For starting points s below the bisector of A and C this can only happen if s is at least vertical distance $d = x\sin 2\alpha$ away from segment AC. For starting points above the the bisector of A and C this can only happen, if s is at least vertical distance $d = x\sin 2\alpha$ away from segment AB. For both cases it remains to consider the points in the rhomboid R. Fortunately, we can use a circle of radius $d' = 0.125$ (slightly larger than d) so that the circle of radius d' with starting point in R touches the boundary with an arc of length at most $2\pi - \tfrac{3}{4}\pi = \tfrac{5}{4}\pi$ and $d'(\tfrac{5}{4}\pi + 1)$ is always strictly smaller than $2x$.
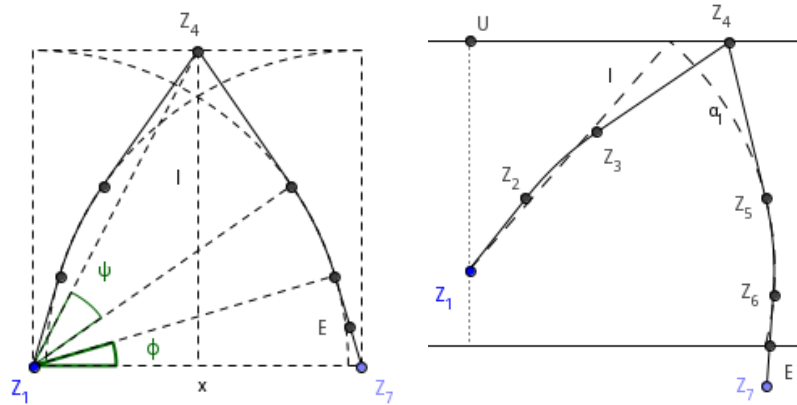

Fig. 7. The figure on the left-hand side shows the curve of ζ. On the right-hand side, ζ has been rotated around Z_1 so that Z_4 lies on the upper bound of the strip. Dashed lines indicate a configuration of the certificate path which is shorter than ζ for the starting position Z_1.

the length of $\Pi_s(l)$ has a maximum value for $d = {}^l\!/_2$ and the length of $\Pi_s(l)$ is slightly below $2.11l$.

We saw that for any starting point in the strip, there are certificate paths for certain distances that are shorter than $2.11l$. As the certificate path is the overall minimum it is below this bound as well. For any starting point, Zalgaller's path ζ can be rotated so that leaving the strip takes at most $2.15l$. Consequently, the certificate path outperforms ζ for any starting point.

The relation to breadth & depth first search. Finally, we would like to relate the certificate to a discrete cost measure Kirkpatrick introduces in [17]. He analyses the problem of digging for oil at m different locations s_i, where $|s_i|$ denotes the (unknown) distance to the source of the oil at the corresponding location. In this scenario, no extra costs arise for switching the location. The challenge is to find a strategy that reaches one source of oil while assuring a small overall digging effort.

At first, Kirkpatrick considers (partially informed) strategies. Those are given all distances from the top to the sources of oil, but not the corresponding location: In case the distances $|s_i|$ have about the same length at all locations, he states that a depth-first searching strategy is certainly effective. Thus, a single location can be chosen for digging, as Figure 8(i) indicates. Although at the chosen location, the distance to the source might be greatest, the digging costs are almost optimal. In case the distance to the source of a single location is significantly shorter than all others, a breadth-first searching strategy performs best. Figure 8(ii) shows that digging at every location with a certain effort x still achieves a small overall effort of $x \cdot m$ in the worst case. These two extreme situations are similar to the cases outlined in Section 3.1 and depicted in Figure 1. For the general case, Kirkpatrick suggests to use a hybrid strategy. If $f_1 \geq f_2 \geq \cdots \geq f_m$ denotes the sorted set of distances, he suggests to choose i so that $i \cdot f_i$ is minimal. The hybrid strategy digs at i (arbitrarily chosen) locations up to the same depth f_i. In the worst case, this strategy reaches a source at the last location with a final effort of $i \cdot f_i$; see Figure 8(iii). Among all such partially informed strategies, this hybrid strategy is certainly optimal and achieves a maximum digging effort of $\lambda := i \cdot f_i$. Similar to this hybrid strategy, we defined the certificate path in the previous section. The certificate path can also be considered as a mixture of depth-first and breadth-first searching. However, the certificate path models a motion. The effort of the digging strategy to explore a certain depth depends on the product of the number chosen locations and the digging depth. In contrast to this, the effort of the certificate path depends on the sum of the searching depth and width. Consequently, the certificate path is a stronger cost measure than the equivalent of the hybrid digging strategy in the plane.

During the further analysis, Kirkpatrick compares a totally uninformed digging strategy to the optimal hybrid strategy. He proves that this strategy approximates the hybrid strategy in $O(\lambda \log(\min(m, \lambda))$ and shows that this factor is tight. Similar to his approach, we compare the certificate path to a totally uninformed spiral strategy and obtain a constant competitive ratio. David Kirk-

patrick [18] brought up the question about what happens in a continuous setting. Note that the game is a quite different in this case, as we take movements in the plane into account and also require a starting orientation.
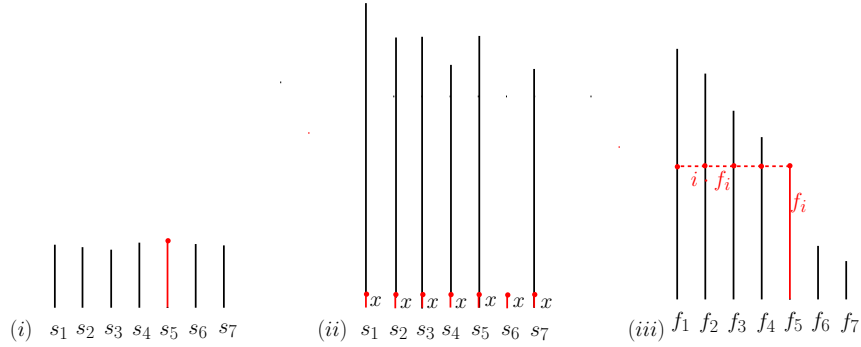

Fig. 8. Online searching for the end of a segment (or digging for oil) for $m = 7$ segments of unknown length. There are two extreme cases: (i) All segments have about the same length. It is reasonable to move along an arbitrary segment up to the end, which is almost optimal. (ii) One segment is significantly shorter than all other segments. One will find the end of a shortest segment by checking all segments with its length. (iii) In case the length of each segment is known, but not the corresponding number of segment. There is always an optimal strategy: Assume that $f_1 \geq f_2 \geq \cdots \geq f_m$ is the decreasing order of the length of all segments. An optimal strategy explores i (arbitrary) segments up to depth f_i, where i is chosen so that $i \cdot f_i = \min_{1 \leq k \leq m} k \cdot f_k$.

4 Online approximation of the certificate path

We are searching for a reasonable escape strategy in an unknown environment. As shown in the previous section, the certificate path and its length is a reasonable candidate for comparisons. Let us assume that $x(1 + \alpha_x)$ is the length of the certificate for some polygon P and for an arbitrary distance x. We can assume that $\alpha_x \in [0, 2\pi]$. This holds since the shortest distance d_s from s to the boundary always results in a candidate $d_s(1 + 2\pi)$. All other reasonable distances x are larger than d_s and $\alpha_x \leq 2\pi$ holds for the optimal x.

Similar to the considerations of Kirkpatrick (see Section 3.2), we would like to guarantee that we leave the polygon P if we have overrun the distance x more than α_x times. This means that the boundary should not wind arbitrarily around s. Therefore, we restrict our consideration to a position s in the kernel of a star-shaped polygon so that there is a single (unknown) distance to the outer boundary in any direction.

In this case we apply the following logarithmic spiral strategy. A logarithmic spiral can be defined by polar coordinates $(\varphi, a \cdot e^{\varphi \cot(\beta)})$ for $\varphi \in (-\infty, \infty)$, a

constant a and an eccentricity β as shown in Figure 9. For an angle ϕ, the path length of the spiral up to point $(\phi, a \cdot e^{\phi \cot(\beta)})$ is given by $\frac{a}{\cos \beta} e^{\phi \cot(\beta)}$. For our
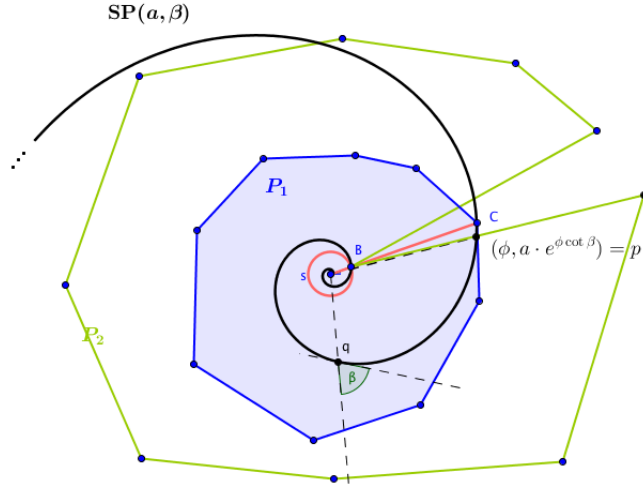

Fig. 9. We apply a spiral strategy for unknown polygons and an unknown starting point s in the kernel. The eccentricity β is chosen so that the two extreme cases have the same ratio. For both polygons P_1 and P_2, the strategy passes the boundary at point $p = (\phi, a \cdot e^{\phi \cot \beta})$ close to C. The path length of the strategy for leaving the polygons is roughly the same. The certificate for P_1 has length $|s\ C|$ (checking the maximal distance to the boundary of P_1), whereas the certificate for P_2 has length $|s\ B|(1 + 2\pi)$ with $|s\ C| = e^{2\pi \cot \beta} |s\ B|$ (checking the smallest distance to the boundary of P_2 with a full circle). We can construct such examples for any point p on the spiral.

purpose we choose β so that the two extreme cases of the certificate attain the same ratio; see Figure 9. We can assume that the certificate of the environment is $x(1 + \alpha_x)$ for an arbitrary distance x and an angle $\alpha_x \in [0, 2\pi]$. Since the spiral strategy checks the distances in a monotonically increasing and periodical way, there has to be some angle ϕ so that $x = a \cdot e^{(\phi - \alpha_x) \cot(\beta)}$ holds. This means that in the worst case, the spiral strategy will leave the environment at point $p = (\phi, a \cdot e^{\phi \cot(\beta)})$ with path length $\frac{a}{\cos \beta} \cdot e^{\phi \cot(\beta)}$. Exactly α_x distances of length x have been exceeded, which means that the boundary has been reached. (Note that, this might not hold for points outside the kernel.)

We would like to choose β so that the two extreme cases $\alpha_x = 0$ and $\alpha_x = 2\pi$ have the same ratio. Thus, we are searching for an angle β so that

$$\frac{\frac{a}{\cos\beta} \cdot e^{\phi\cot\beta}}{a \cdot e^{\phi\cot\beta}(1+0)} = \frac{\frac{a}{\cos\beta} \cdot e^{\phi\cot\beta}}{a \cdot e^{(\phi-2\pi)\cot\beta}(1+2\pi)} \quad \Leftrightarrow \tag{1}$$

$$1 = \frac{e^{2\pi\cot\beta}}{1+2\pi} \tag{2}$$

holds. The right-hand side of Equation (1) shows the case where $x_2 = a \cdot e^{(\phi-2\pi)\cot(\beta)}$ and $\alpha_{x_2} = 2\pi$ gives the certificate and the left-hand side shows the case that $x_1 = a \cdot e^{\phi\cot(\beta)}$ and $\alpha_{x_1} = 0$ gives the certificate $x_i(1 + \alpha_{x_i})$, respectively. In both cases the spiral will detect the boundary latest at point $p = (\phi, a \cdot e^{\phi\cot(\alpha)})$, because the spiral checks 2π distances larger than or equal to x_2 and at least one distance x_1. Figure 9 shows the construction of corresponding polygons P_1 and P_2.

The solution of Equation (2) gives $\beta = \operatorname{arccot}\left(\frac{\ln(2\pi+1)}{2\pi}\right) = 1.264714\ldots$ and the ratio is $\frac{1}{\cos\beta} = 3.3186738\ldots$. Fortunately, for all other values $x = a \cdot e^{(\phi-\gamma)\cot\beta}$ and $\alpha_x = \gamma$ for $\gamma \in (0, 2\pi)$ the ratio is smaller than these two extremes. The overall ratio function is

$$f(\gamma) = \frac{\frac{a}{\cos\beta} \cdot e^{\phi\cot\beta}}{a \cdot e^{(\phi-\gamma)\cot\beta}(1+\gamma)} = \frac{e^{\gamma\cot\beta}}{\cos\beta(1+\gamma)} \quad \text{for } \gamma \in [0, 2\pi] \tag{3}$$

and Figure 10 shows the plot of all possible ratios of the spiral strategy with eccentricity β. Altogether, we have the following result.
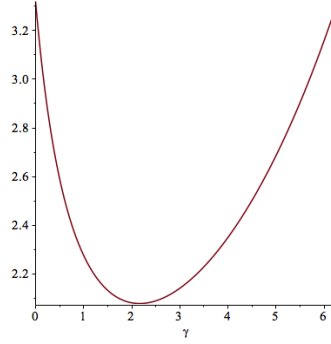

Fig. 10. The graph of the ratio function f of Equation (3) for the spiral strategy with eccentricity $\beta \approx 1.26471$. The two extreme cases 0 and 2π have the same ratio ≈ 3.318674 and all other ratios are strictly smaller.

Theorem 1. *There is a spiral strategy for any unknown starting point s inside the kernel of an unknown environment P that always hits the boundary with path length smaller than* 3.318674 *times the length of the corresponding certificate for s and P.*

Proof. Assume that the certificate of P and s is given by $x(1 + \alpha_x)$. We can set $\gamma := \alpha_x$ and we will also find an angle ϕ so that $x = a \cdot e^{(\phi - \gamma) \cot \beta}$ holds. At point $p = (\phi, a \cdot e^{\phi \cot \beta})$ the spiral has subsumed an arc of angle γ with distances x, so the spiral strategy will leave P at p in the worst case. (Note that, if the start point is not inside the kernel, this might not be true!) The ratio is given by $f(\gamma)$ as in (3) and Figure 10. In the worst case for the strategy γ is either 0 or 2π for the ratio 3.318674, respectively.

We have designed a spiral strategy for some reasonable environments. In the next section, we give a lower bound that shows that this strategy is (almost) optimal for these environments.

Note that a spiral strategy for the online approximation of the certificate path of an arbitrary unknown polygon and position cannot be competitive in general. The polygon might itself wind around the spiral. The ratio against the certificate might be arbitrarily large, consequently. In more general environments other online strategies have to be applied. A potential strategy might be a connected sequence of circles C_i with exponentially increasing radii r^i. This online strategy should result in a constant competitive ratio. Obtaining the optimal strategy for such cases might be complicated and gives rise to future work.

5 Lower bound construction: Online strategy against the certificate

Theorem 2. *Any strategy that escapes from an unknown environment P in unknown position s will achieve a competitive factor of at least* 3.313126 *against the length of a corresponding certificate for s and P in the worst-case.*

Proof. Let us assume that a strategy S is given that attains a better ratio C in the worst case. We consider a bunch of n rays emanating from s with equidistant angle $\frac{2\pi}{n}$ as depicted in Figure 11 for $n = 8$.

The strategy S will successively extend the distances from s also along the rays. Let the sequence $S' = (x_{1,j_1}, x_{2,j_2}, x_{3,j_3}, \ldots)$ describe successive visits of the n rays by the strategy. In x_{i,j_i} the entry i stands for the order and j_i stands for the ray. In S' we only register a visit on ray j_i, if it exceeds the previous visit on the ray j_i. Furthermore, if the distance at ray j_i is exceeded in two successive entries we do not register the first visit in the sequence S'.

In Figure 11(i) we have registered 13 successive visits x_{i,j_i}. Here for example the visit of ray 7 at point q between $x_{9,1}$ and $x_{10,4}$ was not registered in S' because it does not improve the distance of the former visit $x_{7,7}$. Additionally, the visit of ray 1 at point q just before $x_{9,1}$ improved the distance $x_{1,1}$ but it was further improved on $x_{9,1}$ and in between no other ray was improved. For any
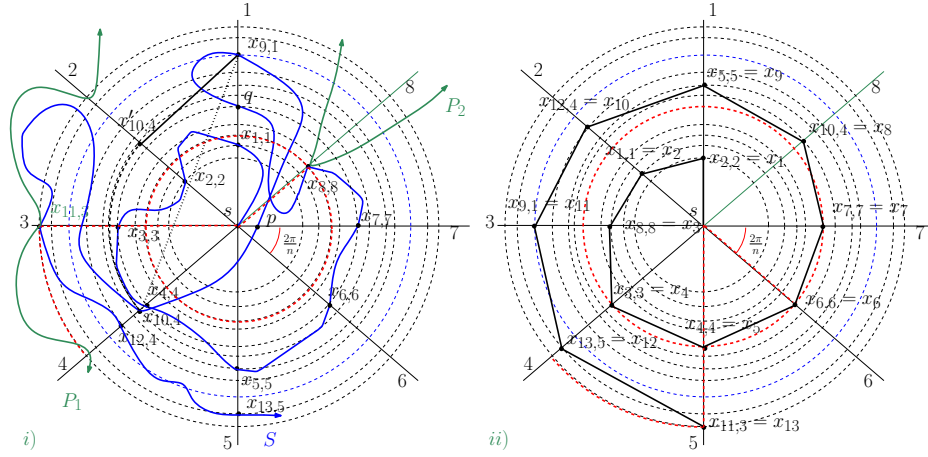
Fig. 11. i) The strategy S results in a sequence S' that represents the visits on n rays successively. There will be a next entry x_{i,j_i} if the strategy exceeds the distance on ray j_i. For two successive extensions on the same ray only the last entry is registered in S'. For the subsequence $S'_{13} = (x_{1,1}, x_{2,2}, \ldots, x_{13,5})$ there will be a last visit on each ray, a minimal distance $x_m = x_{8,8}$ on ray 8 and a maximal distance $x_M = x_{11,3}$ on ray 3. These values gives rise to the construction of certificates for S as sketched by the polygons P_1 and P_2. There are polygons P_1 and P_2 with certificates $x_M(1 + \frac{2\pi}{n})$ and $x_m(1+2\pi)$ and so far S has not been escaped from neither P_1 nor P_2. In S'_{13} the direct distance between two successive points, for example $x_{9,1}$ and $x_{10,4}$, is shorter than the original path length on S and we can further shorten the distance by assuming that $x_{10,4}$ is on the neighbouring ray as depicted by $x'_{10,4}$. ii) We sort the entries of S'_k into a sequence X_k and visit the rays in increasing distance and periodic order. The path length of X_k is not larger than the original path length of S_k and the corresponding certificates for the maximal and minimal value $x'_M = x_M$ and $x'_m > x_m$ are not smaller. Thus, the sum of the corresponding ratios gives a lower bound for the sum of the original ratios.

continuous strategy S we will find such an infinite sequence S'. Let x_{i,j_i} denote the visit and also the distance to the starting point s.

Let us assume that we stop the strategy S at of some ray j_k where the distance was just exceeded on ray j_k, so S' has k steps. Let S_k denote the substrategy of S and S'_k the corresponding subsequence. There will be at least two ratios for S_k that correspond to values of S'_k as follows. In S'_k we consider the last visits on each ray which gives the corresponding maximal visited distance to s for each ray. There will be an overall maximal distance x_M on some ray M_j and a minimal distance x_m on some other ray m_j. In Figure 11(i) we have stopped the strategy S at $x_{13,5}$ on ray 5 and in S'_{13} the minimal distance for the last *round* is given by $x_m = x_{8,8}$ on ray 8 and the maximal distance is given by $x_M = x_{11,3}$ on ray 3.

We can construct polygons P_1 and P_2 so that $x_M(1 + \frac{2\pi}{n})$ is a certificate for P_1 and $x_m(1 + 2\pi)$ is a certificate for P_2. In the first case all other rays have been visited with depth smaller or equal to x_M and we build a polygon P_1 outside S_k that visits any ray at $x_M - \epsilon$ and ray M_j at x_M. This means that a circular strategy with x_M and an arc of length $x_M \frac{2\pi}{n}$ will be sufficient and gives the certificate for P_1 (or at least an upper bound for the certificate of P_1). See for example the polygon P_1 sketched in Figure 11(i) for the maximal visit $x_M = x_{13,5}$. On the other hand for the minimal value x_m we construct a polygon P_2 that hits x_m on ray m_j but runs arbitrarily far away from S_k in any other direction. Thus, $x_m(1 + 2\pi)$ gives the certificate (or at least an upper bound for the certificate of P_2). See for example the polygon P_2 sketched in Figure 11(i) for the minimal visit $x_m = x_{8,8}$. We do not expect that S_k has already detected these polygons but S finally will. So the ratio of the path length $|S_k|$ over $x_m(1 + 2\pi)$ and also the ratio of the path length $|S_k|$ over $x_M(1 + \frac{2\pi}{n})$ give lower bounds for the strategy S. Note that the half of the sum of the two ratios cannot exceed C because otherwise at least one has to be greater than C.

For any such stop we will sort the values of S'_k in a sequence X_k and we will visit the n rays in a monotone and periodic way by sequence X_k connecting the points by line segments; see Figure 11(ii). We can prove that the overall path length of X_k cannot be larger than $|S_k|$.

This can be seen as follows. Successively visiting the points of S'_k in a polygonal chain is already a short cut for S_k. This polygonal path for S'_k might move between two successive values x_{i,j_i} to $x_{i+1,j_{i+1}}$ where j_i and j_{i+1} are not neighbouring rays. In this case we can further short cut the length of the chain of S'_k by just counting a movement from x_{i,j_i} to the distance $x_{i+1,j_{i+1}}$ on one of the directly neighbouring rays. For example in Figure 11(i) the segment from $x_{9,1}$ and $x'_{10,4}$ improves the path length from $x_{9,1}$ and $x_{10,4}$ but passes ray 2 and 3. We only count the distance between $x_{9,1}$ and $x'_{10,4}$ on the neighbouring ray which further improves the length. This means that for a lower bound on the overall path length we can also consider a path that visits two neighbouring rays with angle $\frac{2\pi}{n}$ successively from one to the other with the corresponding depth values x_{i,j_i} stemming from S'_k. By triangle inequality it can be shown that the shortest path that visit all the depth of a sequence S'_k on two rays by

changing from one ray to the other in any step, visits the two rays successively in an increasing order. A similar argument was applied by one author of this article in [20] where a detailed proof of this property is given in the Appendix of [20]. Finally, we can rearrange the path of S'_k to a path that visits the rays in a periodic and monotone way.

Altogether, we have translated the strategy S_k in a discrete strategy $X_k = (x_1, x_2, \ldots, x_k)$ with k entries on n rays that visit the rays in a periodic order such that x_i visits ray $i \bmod n$ and with overall shorter path length; see Figure 11(ii). Consider the corresponding certificates of this new strategy in comparison to the original strategy. For the smallest value on the last round and the largest value on the last round we will obtain a certificate path $x_k(1 + \frac{2\pi}{n})$ which is the same for the previous maximal value $x_M = x_k$ and a certificate path $x_{k-n+1}(1 + 2\pi)$ which is never smaller than $x_m(1 + 2\pi)$ for the minimal value $x_m \leq x_{n-k+1}$. The minimal value can only increase since we sorted the values of S'_k. For example in Figure 11(ii) we have $k = 13$ and the minimal value in the last round is given by $x_6 = x_{6,6}$ which is larger than $x_m = x_{8,8}$. Altogether, the sum of these two ratios in the periodic and monotone setting is always smaller than the sum of the ratios in the original setting.

Finally, we would like to find a periodic and monotone strategy that optimizes the sum of exactly such ratios in this discrete version. This optimal strategy will perform at least as good as any strategy obtained by the above reconstruction. Thus, the optimal value for the sum is a lower bound for the sum of two ratios in the original setting.

For optimizing the sum for an arbitrary strategy we use an infinite sequence of values $X = (x_1, x_2, \ldots)$ and we define the following functionals

$$F_k^1(X) = \frac{\sum_{i=1}^{k-1} \sqrt{x_i^2 - 2\cos\left(\frac{2\pi}{n}\right) x_i x_{i+1} + x_{i+1}^2}}{x_k(1 + \frac{2\pi}{n})} \tag{4}$$

and

$$F_k^2(X) = \frac{\sum_{i=1}^{k-1} \sqrt{x_i^2 - 2\cos\left(\frac{2\pi}{n}\right) x_i x_{i+1} + x_{i+1}^2}}{x_{k-n+1}(1 + 2\pi)} \tag{5}$$

that represent the ratios. We are looking for a sequence X so that

$$\inf_Y \sup_k F_k^1(Y) + F_k^2(Y) = D \text{ and } \sup_k F_k^1(X) + F_k^2(X) = D$$

holds which shows that D is the best sum ratio that we can achieve.

Optimizing such discrete functionals can be done by the method proposed by Gal; see also Gal [13,14], Alpern and Gal [1], and an adaption of Schuierer [24]. It is shown that under certain prerequisites there will be an optimal exponential strategy $x_i = a^i$. The main requirement is that the functional has to fulfil a unimodality property. This means that the piecewise sum of two strategies X and Y is never worse than one of the single strategies. This should also hold for a scalar multiplication of a single strategy. So any linear combination of strategies

that are bounded by a constant will remain bounded by the maximal bound. The proof of Gal shows that in this case we can always combine bounded strategies so that we finally get arbitrarily close to an exponential strategy that has the same bound; see the full proof of Gal in [14] Appendix 2, Theorem 1.

We can easily show that the requirements for the main Theorem of Gal are fulfilled for both functionals $F_k^1(X)$ and $F_k^2(X)$. For a similar functional a detailed proof of this property was given in the Appendix of [20].

Now let us assume that we have an optimal strategy X for the sum, say $F_k^1(X) + F_k^2(X)$. This means that both functionals will also be bounded by constants D_1 and D_2 w.r.t. X. We make use of linear combination of X but apply them independently to the functionals $F_k^1(X)$ and $F_k^2(X)$. The Theorem of Gal shows that we will get arbitrarily close to an exponential strategy $x_i = a^i$ that is not worse than X for both $F_k^1(X)$ and $F_k^2(X)$. This means that $x_i = a^i$ is also not worse than X for the sum functional.

Altogether, it is allowed to search for the best strategy $x_i = a^i$ and we have to optimize

$$\frac{\sum_{i=1}^{k-1} \sqrt{a^{2i} - 2\cos\left(\frac{2\pi}{n}\right)a^{2i+1} + a^{2i+2}}}{a^k\left(1 + \frac{2\pi}{n}\right)} + \frac{\sum_{i=1}^{k-1} \sqrt{a^{2i} - 2\cos\left(\frac{2\pi}{n}\right)a^{2i+1} + a^{2i+2}}}{a^{k-n+1}(1 + 2\pi)}$$

$$\Leftrightarrow \sum_{i=1}^{k-1} a^i \left(\frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right)a + a^2}}{a^k\left(1 + \frac{2\pi}{n}\right)} \right) + \sum_{i=1}^{k-1} a^i \left(\frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right)a + a^2}}{a^{k-n+1}(1 + 2\pi)} \right). \quad (6)$$

For Equation (6) we resolve the geometric serie part and simplify the expression to the minimization of

$$g_n(a) := \frac{1}{a-1}\left(\frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right)a + a^2}}{\left(1 + \frac{2\pi}{n}\right)} \right) + \frac{a^{n+1}}{a-1}\left(\frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right)a + a^2}}{(1 + 2\pi)} \right) \quad (7)$$

We minimize Equation (6) by numerical means. For any number of rays n a minimal value of $g_n(a)$ gives a lower bound on the sum of two ratios in the original problem. So we can choose n as large as we want. We minimize $g_n(a)$ by numerical means using Maple. For example for $n = 28000000000$ we obtain $a = 1.0000000006809\ldots$ and $g(a) = 6.62521\ldots$ This means that for an arbitrary strategy of the original problem there will always be at least one ratio larger than $\frac{6.6252}{2} = 3.313126$ which finishes the proof.

6 Conclusion

We have introduced a new, simple and intuitive performance measure for the comparison against an online escape path for an unknown environment. The measure outperforms the (few) known ultimate optimal escape paths of convex

environments and is also sort of a generalization of a discrete list searching approach by Kirkpatrick.

For a more general class of environments, we presented an online spiral strategy that approximates the measure within an (almost) optimal factor of ≈ 3.318674. Different to classical results, the spiral optimizes against two extremes. It was shown that the factor is almost tight by constructing a lower bound that also holds for arbitrary environments. This is one of the very few cases, where the optimality of spiral search is verified.

Future work might consider randomization. Additionally, it will be helpful to prove the strong conjecture that the certificate path is indeed always better than the shortest escape path for all environments (even when the best the escape path is not known).

Acknowledgements: We would like to thank all anonymous referees for their helpful comments and suggestions.

A Appendix

A.1 Efficient computation of the certificate

In the following, we describe an efficient algorithm to compute Π_s for $s \in P$. We assume that P is bounded by n line-segments. Each segment is defined by two points p_i, p_{i+1} in the plane. Thus, the overall number of points is $n+1$. The algorithm proceeds in two phases. At first, *candidates* for Π_s are computed. For a given radius x, those candidates have a maximum arc α_x in the corresponding direction. The length of such a candidate can be described as a function of the radius x. In a second phase, an efficient algorithm for computing the upper envelope is applied to these functions to obtain the overall shortest path Π_s.

For reasons of simplicity, we assume that a candidate path intersects with the border of P two times. If α_x equals zero or 2π, both intersections are identical. Since we know that a candidate can be described by two segments, we consider its length. Each segment s_i is part of the line $l_i : \{p_i + t \cdot \overrightarrow{p_i p_{i+1}} | t \in \mathbb{R}\}$. We denote d_i the shortest distance from s to a segment s_i and $\angle l_i l_j$ the positive angle between both line segments in the direction of s. Moreover, we denote $[x^-; x^+] \subset \mathbb{R}$ the existence interval for which the candidate path actually intersects both segments on l_i and l_j at the same time. This allows us to express α_x as a function of x

$$\alpha_x = \pi - \angle l_i l_j \pm \left(\arccos\left(\frac{d_i}{r}\right) + \arccos\left(\frac{d_j}{r}\right) \right) \qquad \text{for } x \in \left[x^-; x^+\right].$$

There will not be a candidate path for every pair of segments. As we show in the following, the number of such paths is linear, indeed. We also show how all candidate paths can be computed efficiently.

Lemma 1. *All candidates for the shortest certificate path can be constructed in $O(n \cdot \log n)$ deterministic time. The overall number is bounded by $O(n)$.*

Proof. To compute all candidates, we sweep out the region P with a circle C of expanding radius x around s. Initially, we consider the candidate path touching the segment closest to s. This path always exists and has $\alpha_x = 2\pi$. In the course of the sweep algorithm, the radius x of C expands. Until no other point of the bound of P lies on or in , the initial candidate path is the only one. However, when the circle reaches a segment, the current candidate path also reaches its corresponding existence interval. Depending on how C meets the segment, there are three possible type of events; see Figure 12 on the left.

1. C meets p_i and both neighbouring segments (including p_{i-1} and p_{i+1}) lie inside C. Then, the current candidate path in this direction degenerates to a straight line. We can set x^+ of the candidate path to the current radius of C. The total number of candidates does not change for this event.
2. It may also be the case that C meets p_i so that exactly one of the neighbouring segments lies inside (while the other outside) the circle. Again, the candidate path reaches x^+, as he will no longer intersect with the inner segment. However, there emerges a (single) new candidate path with one intersection on the segment (p_i, p_{i+1}) and the second intersection with the same segment as the current candidate. For this new candidate, we set x^- to the current radius of C. Obviously, the total number of candidates increases by one.
3. The third type of event embraces two cases. Either C meets p_i so that both neighbouring segments lie outside the circle. Or the circle gets tangential to a segment (p_i, p_{i+1}). Both cases involve the creation of two new candidate paths and the limitation of the existence interval of the candidate path in the current direction. The total number of candidates increases by exactly two.

Each type of event involves that one point (or segment) lies on (gets tangential to) the expanding sweep circle C. Afterwards, the point lies inside C- a segment will never get tangential to C, again. As the number of points and segments that define P is linear, the number of events is $O(n)$ for each type of event and all together. This has two consequences.

On the one hand, the sweep algorithm terminates after all $O(n)$ events have been handled. No additional events are created through the course of the algorithm. This gives the running time, as the events have to be handled ordered by x increasing.

On the other hand, each event involves the creation of at most two additional paths; see Figure 12. Thus, the overall number of candidate paths is bounded by $O(n)$.

The length of a candidate path defines a function of a real value x on an existence interval $[x^-; x^+]$. We extend each of these functions to functions over $\mathbb{R}$ by adding a half-line to each end. These half-lines can be chosen to run in parallel to all other additional half-lines. As each length function has a unique maximum or minimum, we can choose the half-lines in a way that they do not intersect more than once with another function. Then, we compute the upper envelope
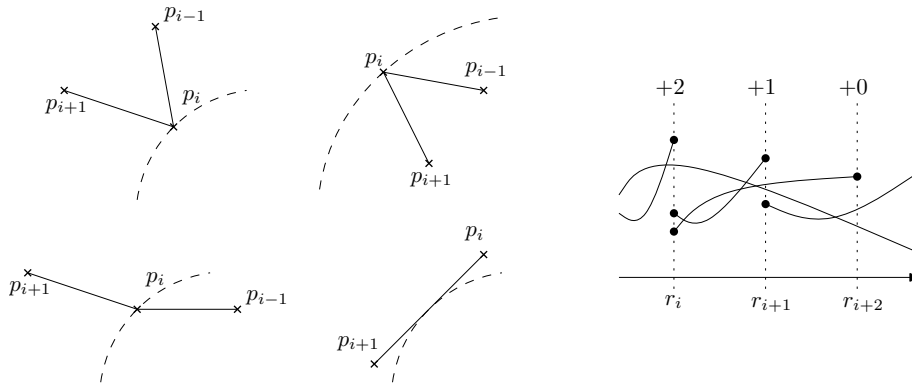
Fig. 12. The figure of on the left shows the different type of events, which may occur when the dashed sweep touches a segment of R. The figure on the right plots the length of several candidate paths as functions of the radius x. At each event r_i, the overall number of candidate paths increases at most by two.

of no more than $O(n)$ curves. Two such curves can intersect only t times for some constant t. Thus, the overall upper envelope of all curves has complexity $O(\lambda_{t+2}(n))$ (almost linear!) and can be computed in $O(\lambda_{t+2}(n) \log n)$; see also Sharir and Agarwal [21].

Having computed the upper envelope of all segments, we check each part of the envelope for a minimum. After $O(\lambda_{t+2}(n))$ such checks, we obtain the certificate path.

Lemma 2. *The certificate Π_s of a polygon P with n edges and a starting point $s \in P$ can be computed in $O(\lambda_{t+2}(n) \log n)$ time for some constant t.*

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