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1.4.2 Competitive ratio of SmartDFS

The corridor of width 3, see Figure 1.7, indicates that the competitive ratio of SmartDFS should be better than 2. SmartDFS runs 4 times though the corridor whereas the shortest path visits any cell only once. This gives roughly a ratio of $\frac{4}{3}$. We will show that this is the worst-case for SmartDFS. The gap between e $\frac{7}{6}$ and $\frac{4}{3}$ is small.

For the analysis we first give a precise definition of the structure of parts of gridpolygons which will be explored in an optimal fashion. The SmartDFS Strategy does not make any detours within these passages.

For a *corridor* of widths 1 this is abviously true. But also corridors of width 2 will be passed optimally, since SmartDFS runs forth and back along different tracks; see Figure 1.17. We give a formal definition of the *narrow passages*.

Definition 1.14 The set of cells that can be deleted such that the layernumber of the remaining cells do not change are called narrow passages of P.

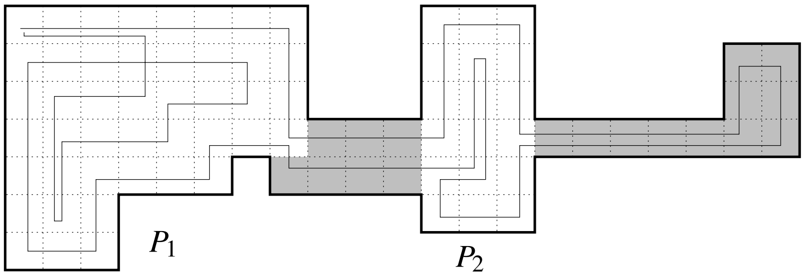

Figure 1.17: SmartDFS is optimal in narrow passages.

SmartDFS passes narrow passages optimally since they allow an optimal forth and back pass-through. There are no additional detours at the entrance and exit of a narrow passage because they consist of cells in the first layer. They can be considered as *gates*. The entrance and exit is always precisely determined.

The idea is to consider polygons without narrow passages first. There is a fixed relationship between edges and cells.

Lemma 1.15 *Let P be a simple gridpolygon without narrow passages and without a split-cell in the first layer. We have*

$$E(P) \leq \frac{2}{3}C(P) + 6\,.$$

Proof. A 3×3 gridpolygon has precisely this property, $C(P) = 9$ und $E(P) = 12$. Any gridpolygon with the above conditions can be reduced by successively removing columns or rows such that in each step the property remains true and such that always at least 3 cells and at most 2 edges will be removed. This is an exercise below.

Starting backwards from the property $E(P_0 = \frac{2}{3}C(P_0) + 6$ we will maintain the bound $E(P_i) \leq \frac{2}{3}C(P_i) + 6$ since we add at least 3 cells and add at most 2 edges. Finally, $E(P) \leq \frac{2}{3}C(P) + 6$ holds. □

First, we show that the overall number of exploration steps of SmartDFS decreases for the given class of polygons.

Lemma 1.16 *A simple gridpolygon P with $E(P)$ edges and $C(P)$ cells, without narrow passages and without a split-cell in the first layer well be explored by SmartDFS with no more than $S(P) \leq C(P) + \frac{1}{2}E(P) - 5$ steps.*

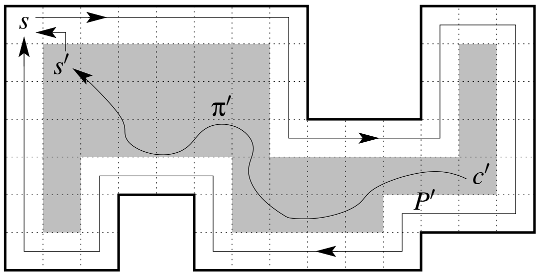

Figure 1.18: A simple gridpolygon without narrow passages and no split-cell in the first layer has the property $E(P) \leq \frac{2}{3}C(P)+6$. After the first coil SmartDFS starts in the 1-Offset P'. The return path to c' from an arbitrary point in P' is shorter than $\frac{1}{2}E(P)/2-2$.

Proof. From Theorem 1.13 we conclud $S(P) \leq C(P)+\frac{1}{2}E(P)-3$. By the properties of P, SmartDFS performs a full first round from s to the first cell s' in the second layer. After that, in principle we start SmartDFS again at s' in a gridpolygon (1-Offset P' of P); see P' in Figure 1.18. P' is path connected and by Lemma 1.9 P' has 8 edges less than P;

The cells in the first layer have been visited optimally the path length from s to s'coincidence with the number of cells in the first layer. Finally, we have to count two additional steps from s' to s. Altogether, we require $S(P) \leq C(P)+\frac{1}{2}(E(P)-8)-3+2 = C(P)+\frac{1}{2}E(P)-5$ steps. □

With the statements above we will be able to prove the main result.

Mit diesen Vorbereitungen können wir die kompetitive Schranke beweisen.

Theorem 1.17 *(Icking, Kamphans, Klein, Langetepe, 2005) The SmartDFS strategie for the exploration of simple gridpolygons is $\frac{4}{3}$-competitive!* *[IKKL05]*

Proof.

Let P be a simple gridpolygon. First, we remove the narrow passages from P. We know that the entrance and exits over the gates by SmartDFS are optimal. We obtain a sequence P_i, $i = 1,\dots,k$ of gridpolygons connected by narrow passages. See for example P_1 and P_2 in Figure 1.17.

We can consider the gridpolygons P_i separately. We can also assume different starting points. The movement between the gates count for the required additional steps. It is sufficient to show $S(P_i) \leq \frac{4}{3}C(P_i)-2$ for any subpolygon. This bound exactly holds for $3\times m$ gridpolygons for even m; see Figure 1.19.

We show the bound by induction over the number of splil-cells.

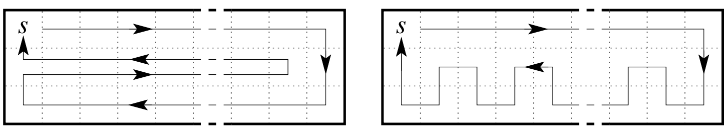

Figure 1.19: In a corridor of width 3 and with even lenght the bound $S(P) = \frac{4}{3}S_{\text{Opt}}(P)-2$ holds.

Induktion-Base: If P_i has no split-cell, there is also no split-cell in the first layer. We apply Lemma 1.16 and Lemma 1.15 and obtain:

$$S(P_i) \quad \leq \quad C(P_i)+\frac{1}{2}E(P_i)-5$$

$$\begin{aligned} &\leq C(P_i) + \frac{1}{2}\left(\frac{2}{3}C(P_i) + 6\right) - 5 \\ &= \frac{4}{3}C(P_i) - 2\,. \end{aligned}$$

Induktion-Step: If there is no split-cell in the first layer we can apply the same arguments as above. Therefore, we assume that the first split occurs in the first layer. Two cases can occur as depicted in Figure 1.20.

In the first case the component of type (II) was not visited before and we define $Q := \{c\}$. The second case occurs, if the split-cell c is diagonally adjacent to a cell c'; compare Figure 1.20(ii), (iii) and (iv). We build the smallest rectangle Q that contains c and c'. In case (ii) and (iii) Q is a square of size 4. In case (iv) by simple adjacency Q is a rectangle and $|Q| = 2$.

Analogously to the proof of Theorem 1.13 we split the polygons into parts P' and P'' both containing Q.

Here P'' is of type (I) or (II) and P' the remaining polygon. das Polygon der Komponenten vom Typ (I) oder (II) und P' das andere.

For $|Q| = 1$ (see Figure 1.20(i)) we have $S(P_i) = S(P') + S(P'')$ and $C(P_i) = C(P') + C(P'') - 1$. We apply the induction hypothesis on P' and P'' (they have one split-cell less) and obtain:

$$\begin{aligned} S(P_i) &= S(P') + S(P'') \\ &\leq \frac{4}{3}C(P') - 2 + \frac{4}{3}C(P'') - 2 \\ &\leq \frac{4}{3}C(P_i) + \frac{4}{3} - 4 < \frac{4}{3}C(P_i) - 2\,. \end{aligned}$$

For $|Q| = 4$ we argue that by the union we will save some steps that will occur for the separate explorations. We consider P' and P'' separately, first. The movements from c' to c (and c to c') count in both polygons. For the complete P_i the path from c' to c (and c to c') are given either P' or in P'', this means that we save $4 = |Q|$ steps.

We have $S(P_i) = S(P') + S(P'') - 4$ and $C(P_i) = C(P') + C(P'') - 4$. By induction hypothesis for P' and P'' we conclude:

$$\begin{aligned} S(P_i) &= S(P') + S(P'') - 4 \\ &\leq \frac{4}{3}C(P') + \frac{4}{3}C(P'') - 8 \\ &= \frac{4}{3}\left(C(P') + C(P'') - 4\right) - \frac{8}{3} \\ &< \frac{4}{3}C(P_i) - 2\,. \end{aligned}$$

The case $|Q| = 2$ is left as an exercise.

Altogether an optimal strategy requires $\geq C(P_i)$ steps or $\geq C(P)$ in total and we have a competitive ratio of $\frac{4}{3}$. □

Exercise 8 *Analyse the remaining case $|Q| = 2$ in the above proof.*

If we compare the result to Theorem 1.7 there is a gap of size $\frac{1}{6}$ between $\frac{7}{6}$ and $\frac{4}{3}$. Recently, both parts have been improved. There is a lower bound of $\frac{20}{17}$ and an upper bound of $\frac{5}{4}$ shown by Kolenderska et. al 2010. In principle the strategy is a local improvement of SmartDFS and the lower bound is an extension of our construction. The result comes along with a tedious case analysis.