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Online Motion Planning

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Chapter 3

Online searching for objects

In this chapter we collect several results that consider the task of searching for an unknown object. Different from the navigation task, the position of the object is not known. The object is detected due to the sensor abilities of the agent.

For example one might consider the case that the agent is equipped with a sight system. For example inside simple polygons we can assume that the agent is point-shaped and the visibility information is given by the visibility polygon.

Definition 3.1 Let P be a simple polygon and r a point with $r \in P$. The **visibility polygon** of r w.r.t P, $\text{Vis}_P(r)$, is the set of all points $q \in P$, that are visible from r inside P, i.e., the line segment $\overline{rq}$ is fully inside P.

We start our consideration by searching for a goal inside a corridor polygon. The polygon can also be modelled by a line. Note that the visibility information is not helpful, if the corridor has many small kinks or caves, where the goal might be hidden.

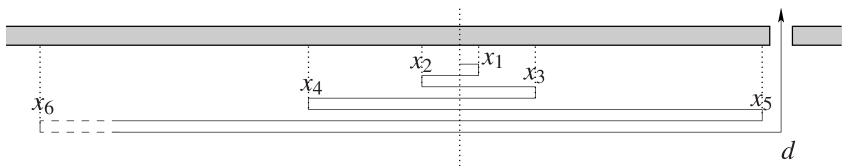

Figure 3.1: Searching for a door along a line.

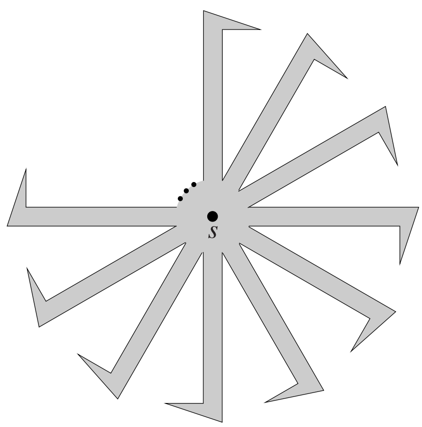

Figure 3.2: In which corridor lies the target point?

3.1 2-ray search and the Theorem of Gal

As a special case we consider the problem of searching for a point along a line (or on 2 rays emanating from a starting point). This problem is also known as the lost-cow or cow-path problem; see citekrt-sueor-93. The cow is searching for the hole in the fence or an agent is searching for a door along a wall. Form the starting position neither the direction nor the distance d to the goal (door/hole) is given. We assume that the agent has no sight system or as mentioned above at any point on the line the sight is very limited (for example by many kinks). Thus the agent detects the goal only by visiting the goal exactly.

The agent cannot concentrate on one direction, because the goal might be on the opposite side. Any reasonable strategy successively changes the direction and can be described by an infinite sequence $(x_1, x_2, x_3, \ldots)$ with $x_i > 0$. The agent runs x_1 steps to the right, returns to the start, runs x_2 steps to the left, returns to the start, runs $x_3 > x_1$ steps to the right again and so on; see Figure 3.1. For the searching depth x_j we can assume that they are monotonically increasing on each side, i.e., $x_{i+2} > x_i$.

We compare the length of the agents path to the shortest path to a goal. It is sufficient to consider the local worst-case situation. In the beginning the agent returns to the start by path of length $2x_1$. If the goal is located arbitrarily close on the opposite side, there is no competitive strategy since for any $C > 0$ there will be an $\varepsilon > 0$ such that $2x_1 + \varepsilon > C \cdot \varepsilon$ holds. Therefore we require an additive constant in this case. Alternatively, we assume that the goal is at least step 1 away from the start. These two interpretation are equivalent.

Exercise 18 *Show that for the 2-ray search problem, the following interpretations are equivalent: A strategy is C competitive with some additive constant A. A strategy is C competitive without additive constant but the goal is at least step B away from the start.*

We assume that the goal is at least one step away from the start. The local worst-case for the competitive ratio is that we slightly miss the target at distance d by an ε, perform another turn on the opposite side and met the target at distance $d + \varepsilon$; see Figure 3.1. Now the task is to find a sequence $(x_1, x_2, x_3, \ldots)$ such that

$$\sum_{i=1}^{k+1} 2x_i + x_k \leq C \cdot x_k$$

holds for all k and C is as small as possible. This means that

$$\frac{\sum_{i=1}^{k+1} 2x_i + x_k}{x_k} = 1 + 2\frac{\sum_{i=1}^{k+1} x_i}{x_k}$$

has to be minimized for all k.

A reasonable stratagy doubles the distance all the time, that is $x_i = 2^i$; see Figure 3.1. Such a doubling heuristic is indeed optimal as shown in [BYCR93] and also in [Kle97]. The competitive ratio is bounded by $C = 9$.

Form $\sum_{i=1}^{k+1} x^i = \frac{x^{k+2}-1}{x-1} - 1$ we conclude $\frac{\sum_{i=1}^{k+1} 2^i}{2^k} = \frac{2^{k+2}-2}{2^k} = 4 - \frac{2}{2^k} < 4$ and attain 9 as the overall ratio which is attained asymptotically.

In the context of Search Games Gal [Gal80] has shown that under certain condition such functionals $F_k(f_1, f_2, \ldots, f_{k+1}) := \frac{\sum_{i=1}^{k+1} f_i}{f_k}$ can be minimized for all k by an exponential sequence $f_i = a^i$ for some $a > 1$. Here $X = (f_1, f_2, \ldots)$ is a sequence of positive values f_i and the functional F_k depends on $k+1$ successive entries of X. We can also write $F_k(f_1, f_2, \ldots)$ statt $F_k(f_1, f_2, \ldots, f_{k+1})$ it the number of entries used is clear from the context. We are searching for an optimal sequence X. More precisely: The supremum of $F_k(X)$ over alle k gives the perfomance of X (it need not be a maximum) and we are searching for a sequence X that gives the infimum on all such suprema (it need not be a minimum). Thus we are searching for X such that

$$\inf_Y \sup_k F_k(Y) = C \text{ und } \sup_k F_k(X) = C.$$

The following general result helps to optimize functionals in the sense explained above. Ee do not give a formal proof for it.

Theorem 3.2 *(Gal, Alpern, 2003)*
Given a sequence of functionals $F_k(X)$ for all $k \geq k_0$ and a sequence $X = (x_1, x_2, x_3, \dots)$ with $x_i > 0$. For X let $k+i$ be the largest Index, so that $F_k(X)$ depends on x_{k+i}.

For two sequences $X = (x_1, x_2, x_3, \dots)$ and $Y = (y_1, y_2, y_3, \dots)$ let $X + Y := (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots)$ and $\alpha \cdot X := (\alpha \cdot x_1, \alpha \cdot x_2, \alpha \cdot x_3, \dots)$.

If F_k fulfils the conditions:

(i) *F_k ist stetig,*

(ii) *F_k ist unimodal: $F_k(\alpha \cdot X) = F_k(X)$ und $F_k(X+Y) \leq \max\{F_k(X), F_k(Y)\}$,*

(iii)

$$\lim_{a \mapsto \infty} \inf F_k \left(\frac{1}{a^{k+i}}, \frac{1}{a^{k+i-1}}, \dots, \frac{1}{a}, 1 \right) = \lim \inf_{\varepsilon_{k+i}, \varepsilon_{k+i-1}, \dots, \varepsilon_1 \mapsto 0} F_k \left(\varepsilon_{k+i}, \varepsilon_{k+i-1}, \dots, \varepsilon_1, 1 \right),$$

(iv)

$$\lim_{a \mapsto 0} \inf F_k \left(1, a, a^2, \dots, a^{k+i} \right) = \lim \inf_{\varepsilon_{k+i}, \varepsilon_{k+i-1}, \dots, \varepsilon_1 \mapsto 0} F_k \left(1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k+i} \right),$$

(v) *$F_{k+1}(f_1, \dots, f_{k+i+1}) \geq F_k(f_2, \dots, f_{k+i+1})$.*

Then

$$\sup_k F_k(X) \geq \inf_a \sup_k F_k(A_a)$$

where $A_a = a^0, a^1, a^2, \dots$ und $a > 0$. *[AG03]*

The Theorem says that due to the correctness of some natural conditions the functionals F_k can be minimized for all k by an exponential sequence. We can apply the Theorem on the 2-ray search problem and have to show that the requirements hold for $F_k(f_1, f_2, \dots) := \frac{\sum_{i=1}^{k+1} f_i}{f_k}$. Obviously, F_k is continuous. Also unimodality holds. We have
$\frac{\sum_{i=1}^{k+1} \alpha \cdot f_i}{\alpha \cdot f_k} = \frac{\sum_{i=1}^{k+1} f_i}{f_k}$. Additionally, from the general statement $\frac{a}{b} \geq \frac{c}{d} \Leftrightarrow \frac{a+c}{d+b} \leq \frac{a}{b}$ (shown by simple equivalence) we conclude that for two sequences $X = (x_1, x_2, x_3, \dots)$ and $Y = (y_1, y_2, y_3, \dots)$ we have $F_k(X+Y) \leq \max\{F_k(X), F_k(Y)\}$. The corresponding limits for the special sequences $(\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_1, 1)$ and $(1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$, respectively run to infinity and the statemens (iii) and (iv) trivially hold. Also statement (v) holds because on the left hand side there is an additional number in the numerator. Altogether, $\frac{\sum_{i=1}^{k+1} f_i}{f_k}$ is minimized by $f_i = a^i$. Therefore, the remaining task is to find the best value for a. From $\sum_{i=1}^{k+1} a^i = \frac{a^{k+2}-1}{a-1} - 1$ and the fact that a^k has to increase arbitrarily we conclude that $f(a) := \frac{a^2}{a-1}$ has to be minimized. By analytic means we obtain $a = 2$ as the optimal value for minimization.

3.1.1 Generalization to m-rays

We can easily extend the problem by considering m corridors emanating from a common starting point which gives the so-called m-ray-search problem; see Figure 3.2. If we consider the situation as searching for a target point inside a polygon, we can already state that this problem is not competitive in general.

Assume that the caves at the end of the m corridors have distance 1 from the start. The agent has to look inside any cave and the adversary places the target in the last cave. This gives a path length of $2(m-1)+1$ versus the shortest path of length 1. So for any C there exists a polygon such that no strategy can guarantee a competitive ratio smaller than C. We just choose the m-corridor polygon for $C < 2(m-1)+1$. This means that the optimal competitive ratio for m rays should depend on m.

So we consider a fixed m and m rays that emanate from a common starting point s. The agent has no sight system and detects the goal only by an exact visit. We make use of the following notations. By f_j we denote the j-th step where the agent visits some ray at depth f_j. Let J_j denote the index of the *next* visit of this ray. We require this index for describing the local worst case situation. So let (f_j, J_j) be the corresponding pairs for all j. Now the performance of a strategy $(f_1, f_2, \dots)$ obviuously is given by

$$\sup_l \left(1 + 2 \frac{\sum_{i=1}^{J_l - 1} f_i}{f_l} \right).$$

In order to apply the Theorem of Gal we assume that the rays are visited in a periodic order ($J_j = j+m$) and with overall increasing depth ($f_j \leq f_{j+1}$). Figure 3.3 shows an example with some steps for $m = 4$. Now we have to minimize $F_k(f_1, f_2, \ldots) := \frac{f_k + 2\sum_{i=1}^{k+m-1} f_i}{f_k}$ over all k.

By the same arguments as before F_k fulfills the requirements of Theorem 3.2 and we conclude $\sup_k F_k(X) \geq \inf_a \sup_k F_k(A_a)$ for $A_a = a^0, a^1, a^2, \ldots$ and $a > 0$. In analogy to the 2-ray case we attain a function $f(a)$ independent from k and $f(a)$ has to be minimized. This optimization is a given as an exercise.

Exercise 19 *Minimize the functionals $F_k(f_1, f_2, \ldots) := \frac{f_k + 2\sum_{i=1}^{k+m-1} f_i}{f_k}$ and show that the (optimal) competitive ratio of the m-ray search problem for periodic and monotone strategies is $C = 1 + 2m\left(\frac{m}{m-1}\right)^{m-1}$. An optimal exponential strategy a^i for this case is given by $a = \frac{m}{m-1}$.*

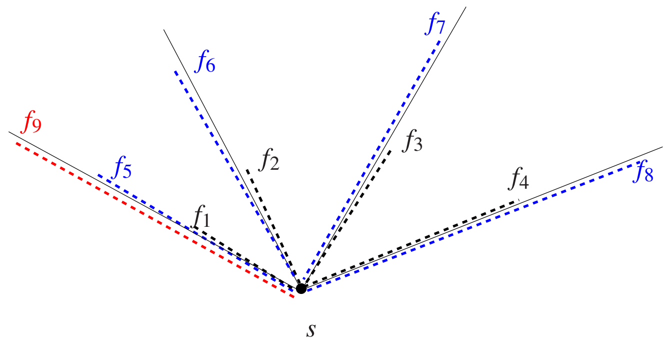

Figure 3.3: The first steps of a periodic and monotone strategy for $m = 4$ rays.

The above optimization was a simple application of Theorem 3.2. The main problem is that we assumed that there is an optimal strategy that is periodic and monotone. This is an instance of a more general problem. There are some motion planning problems where the existence of periodic and monotone optimal solutions is not known. The best upper bounds for competitive ratios are often achieved by the above assumption. Lower bounds are much harder to achieve. For the m-ray configuration the existence of periodic and monotone optimal solutions can be shown.

Lemma 3.3 *There is always an optimal competitive (with the best overall achievable ratio C) m-ray search strategy $(f_1, f_2, \ldots)$, that visits the rays in overall increasing depth and in periodic order.*

Proof. Assume that there is an arbitrary C-competitive strategy $(f_1, f_2, \ldots)$ with pairs (f_j, J_j) for the smallest attainable ratio C. First, we show that we can rearrange this strategy to a monotone C-competitive strategy $(f_1', f_2', \ldots)$, i.e., $f_j' \leq f_{j+1}'$ holds for all j.

Let j be the smallest index j such that $f_j > f_{j+1}$. The performance $(C-1)/2$ for the strategy $(f_1, f_2, \ldots)$ is represented in

$$\sum_{i=1}^{J_j - 1} f_i \leq \frac{C-1}{2} f_j \quad : \quad \text{for index } j \tag{3.1}$$

$$\sum_{i=1}^{J_{j+1} - 1} f_i \leq \frac{C-1}{2} f_{j+1} \quad : \quad \text{for index } j+1 \tag{3.2}$$

$$\sum_{i=1}^{J_l - 1} f_i \leq \frac{C-1}{2} f_l \quad : \quad \text{for index } l \neq j, j+1 \tag{3.3}$$

We exchange f_j and f_{j+1} by $f_j' := f_{j+1}$ and $f_{j+1}' := f_j$. What happens to the performance above? Inequality (3.2) remains true, because we only increase the right hand side. If $J_{j+1} > J_j$ holds, also the first inequality (3.2) is maintained, because the original second inequality (for f_{j+1} on the right hand side) hold and the left hand side of inequality (3.1) is even smaller now. The remaining inequlities (3.3) are not concerned from this exchange.

The remaining task is to handle the case $J_{j+1} < J_j$. Here we have the problem of maintaining inequality (3.1). To overcome this problem we exchange the role of the rays of f_j und f_{j+1} directly after the index $j+1$ completely. After index $j+1$ any original visit of the ray of f_{j+1} is no applied to the ray of f_j and vice versa. Of course the exchange $f'_j = f_{j+1}$ and $f'_{j+1} = f_j$ is maintained. Now we do not have a problem with inequality (3.1) any more since the ray is visited early enough now. Inequality (3.2) is also maintained because we have the same next visits as before. Inequalities (3.3) do not change, they are not influenced by the exchange. In principle for $J_{j+1} < J_j$ and $f_j > f_{j+1}$ we exchange two complete rays beginning with index j.

For example if $f_1 > f_2$ holds and $J_1 = 7$ for ray K and $J_2 = 5$ for ray L, then after the exchange we visit K by $f'_1 := f_2$ then L by $f'_2 := f_1$, later K by $f'_5 := f_5$ and L by $f'_7 := f_7$ and so on. Figure 3.4 shows an example.

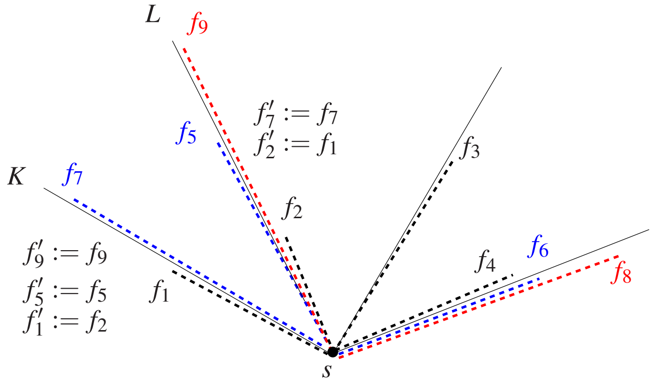

Figure 3.4: A non-periodic and non-monotone strategy. First, we exchange the values f_1 and f_2 only. But since $J_1 = 7 > J_2 = 5$ holds we fully exchange the role for the corresponding rays K and L.

Altogether, we obtain a C-competitive strategy $(f'_1, f'_2, \ldots)$ with $f'_j \leq f'_{j+1}$ for all j by applying the above exchange successively.

Finally, we construct a periodic strategy by the same idea. Consider a monotone strategy with a first index j such that $J_{j+1} < J_j$. We exchange the role of the corresponding rays after step $j+1$, which means that f_j and f_{j+1} remain on their place. Now $J'_{j+1} > J'_j$ holds. The ray with smaller f_j is visited earlier which maintains the ratio, the ray with next at visit J_j s visited later now but the original strategy maintains the ratio for the corresponding sum with f_j and we have f_{j+1} on the right hand side now. All other inequalities are not concerned.

Now after this change it might happen that some the monotonicity after step $j+1$ is no longer given. Then we apply the first rearrangement again and so on.

Altogether we obtain a monotone strategy with $J_{j+1} > J_j$ for all j and the same ratio C. Trivially, if $J_{j+1} > J_j$ holds for all j, this can only mean that $J_j = j + m$ holds for all j. □

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