# Voronoi Diagram and Delaunay Triangulation

## Randomized Incremental Construction

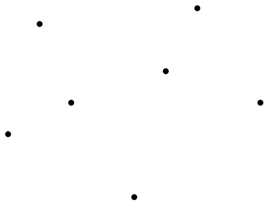
Chih-Hung Liu

May 13, 2015

# Outline

# Voronoi Diagram

- Given a set $S$ of $n$ point sites, Voronoi Diagram $V(S)$ is a planar subdivision



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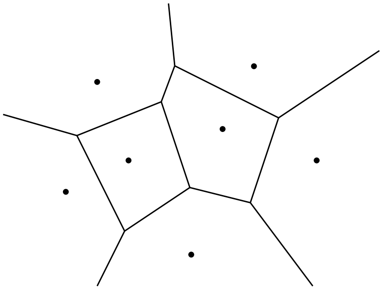
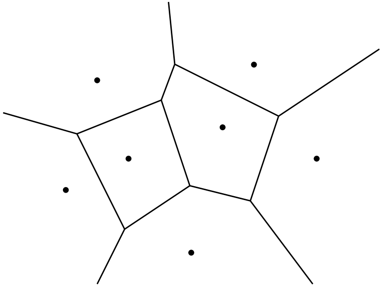


# Voronoi Diagram

- Given a set $S$ of $n$ point sites, Voronoi Diagram $V(S)$ is a planar subdivision
  1. Each region contains exactly one site $p \in S$ and is denoted by $\mathrm{VR}(p, S)$.



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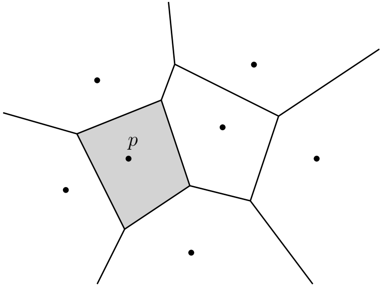

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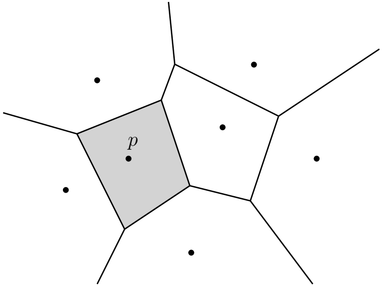

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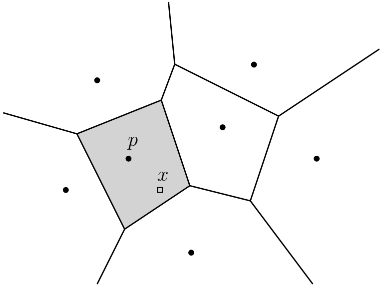

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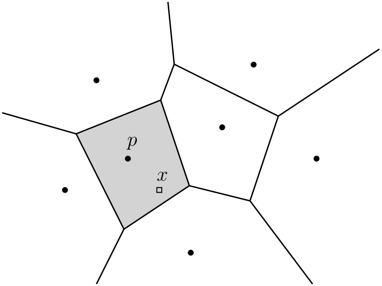
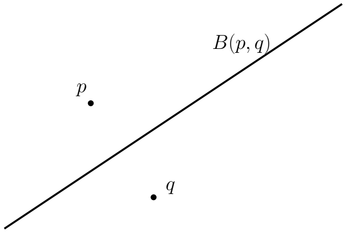
  1. Each region contains exactly one site $p \in S$ and is denoted by $\mathrm{VR}(p, S)$.
  2. For each point $x \in \mathrm{VR}(p, S)$, $p$ is its closest site in $S$.
- $\mathrm{VR}(p, S)$ is the locus of points closer to $p$ than any other site.



# Voronoi Region

- Bisector $\boldsymbol{B(p,q)} = \{x \in R^2 \mid d(x,p) = d(x,q)\}$.



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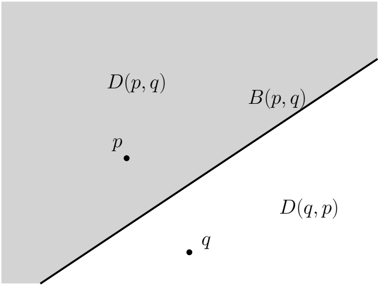
  - Two half-planes $D(p,q)$ and $D(q,p)$ separated by $B(p,q)$.



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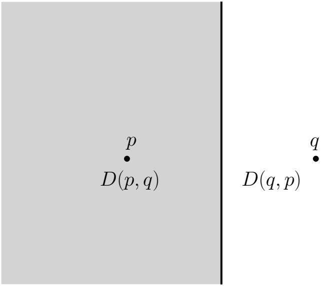

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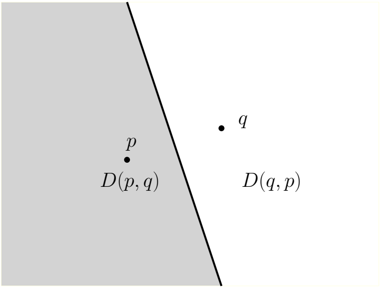

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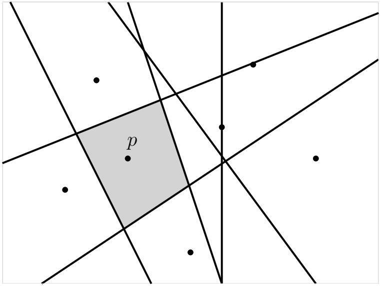
  - Two half-planes $D(p, q)$ and $D(q, p)$ separated by $B(p, q)$.
- $$\mathrm{VR}(p, S) = \bigcap_{q \in S, q \neq p} D(p, q).$$



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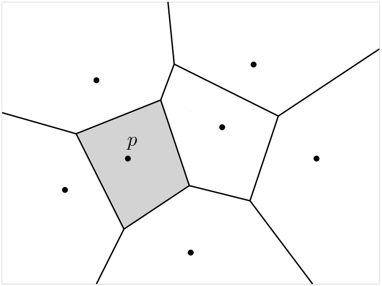
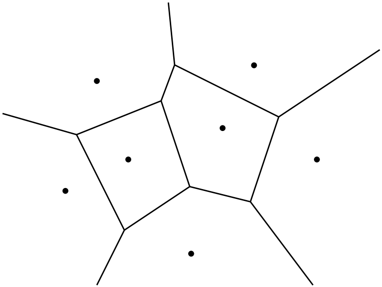


# Voronoi Edge and Vertex

- Voronoi Edge
  - Common intersection between two adjacent Voronoi regions $VR(p, S)$ and $VR(q, S)$



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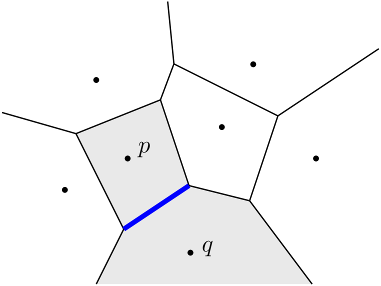

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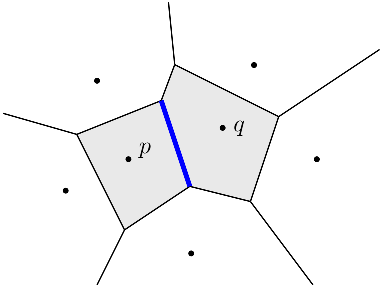

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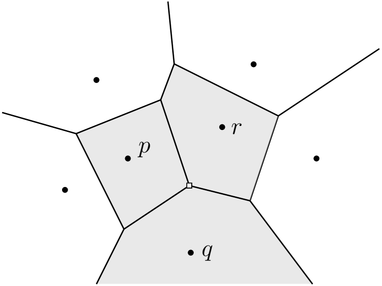
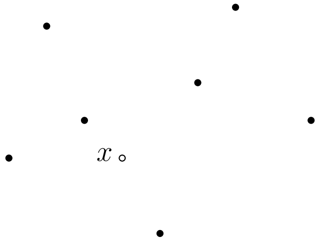
  - A piece of $B(p, q)$
- Voronoi Vertex
  - Common intersection among more than two Voronoi regions $\mathrm{VR}(p, S)$, $\mathrm{VR}(q, S)$, $\mathrm{VR}(r, S)$, and so on.



# Growing Circle

- Grow a circle from a point *x* on the plane



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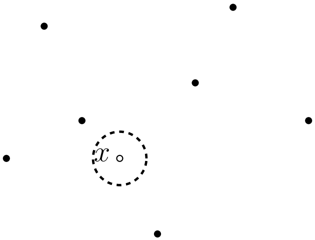

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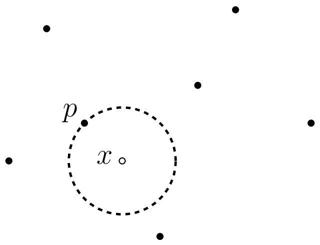

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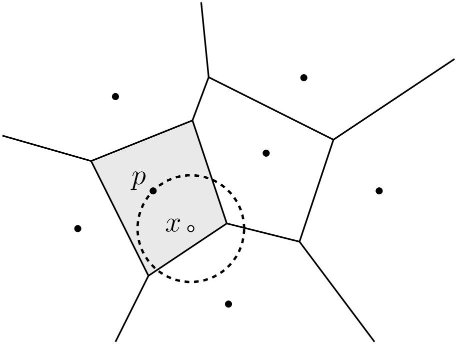

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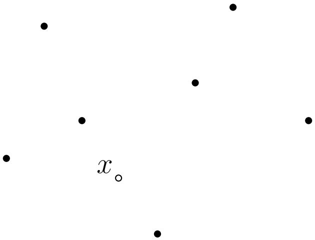

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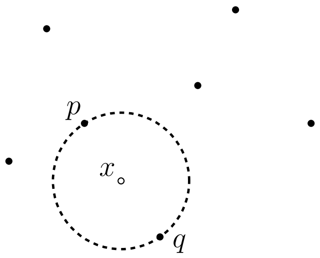

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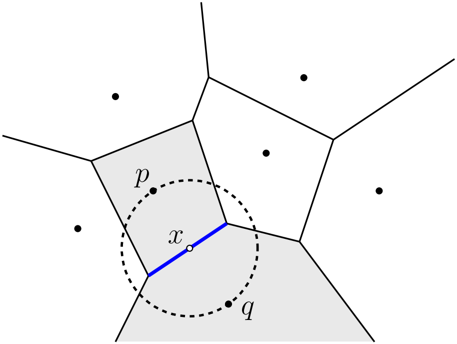

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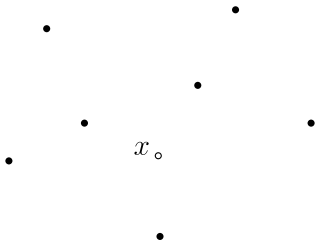

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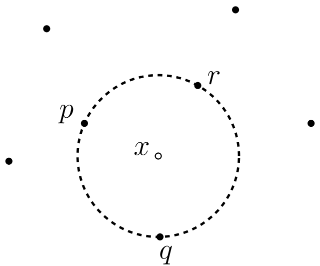

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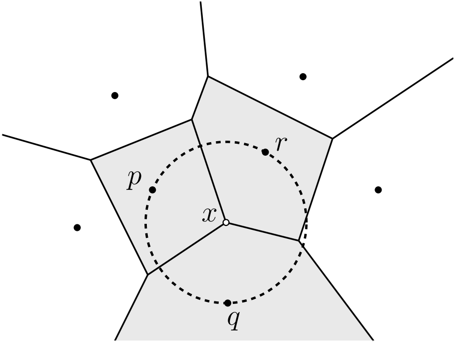
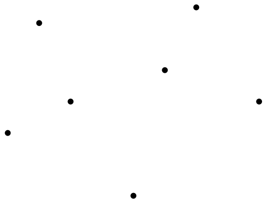


# Wavefront Model (Growth Model)

- Grow circles from $\forall p \in S$ at unit speed



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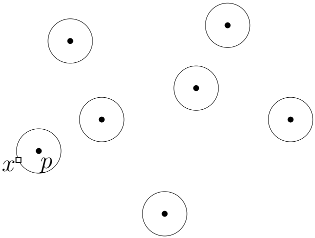

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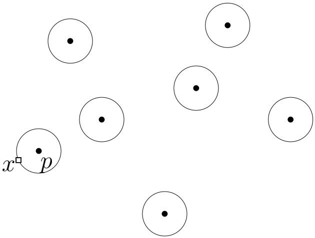

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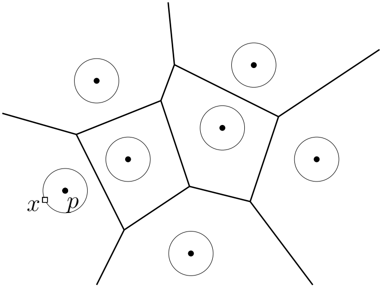

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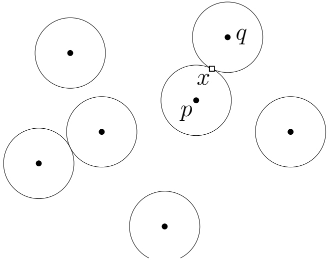

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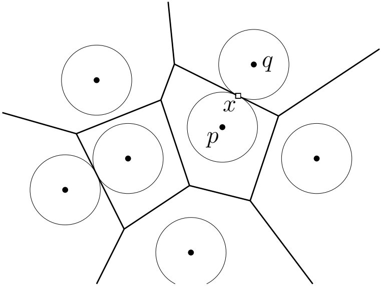

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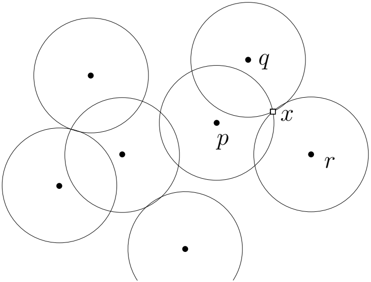

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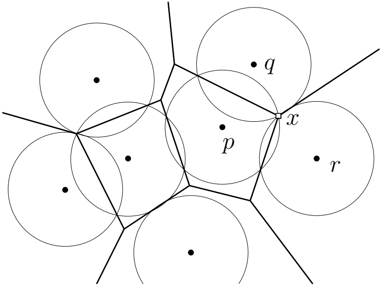
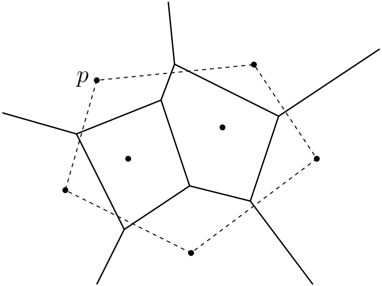


# Unbounded Region

- VR($p$, $S$) is unbounded if and only if $p$ is a vertex of the convex hull of $S$.



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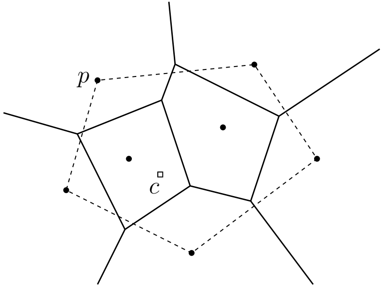

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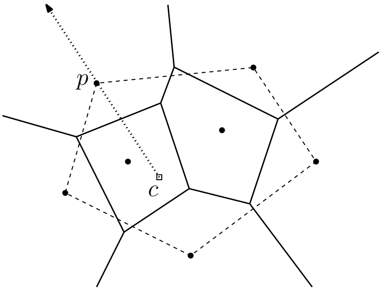

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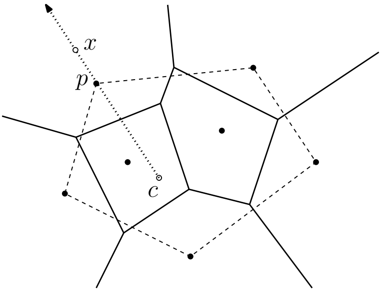
  - Select a point $c$ in the convex hull
  - Shoot a ray $\overrightarrow{cp}$ from $c$ to $p$
  - For any point $x \in \overrightarrow{cp} \setminus \overline{cp}$, $x$ belongs to VR($p$, $S$)



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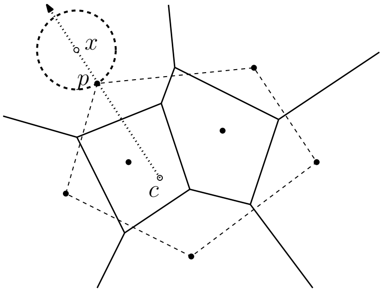

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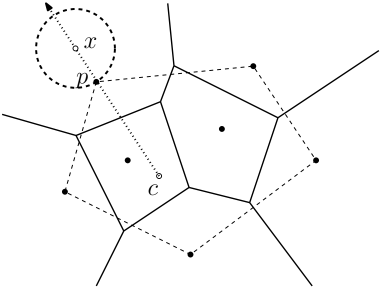
  - $\overrightarrow{cp}$ extends to the infinity.



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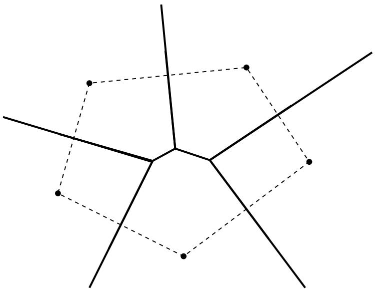
  - $\overrightarrow{cp}$ extends to the infinity.
- If $S$ is in convex position, $V(S)$ is a tree.



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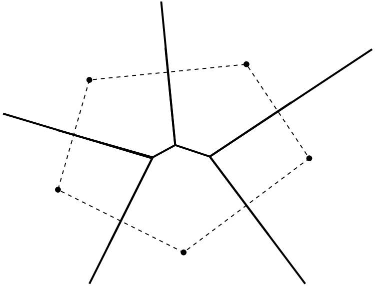
- If $S$ is in convex position, $V(S)$ is a tree.
- An unbounded Voronoi edge corresponds to a hull edge.



# Voronoi Diagram (Mathematic Definition)

- Voronoi Diagram $V(S)$

$$V(S) = R^2 \setminus (\bigcup_{p \in S} \mathrm{VR}(p, S)) = \bigcup_{p \in S} \partial \mathrm{VR}(p, S)$$

  - $\partial \mathrm{VR}(p, S)$ is the boundary of $\mathrm{VR}(p, S)$
    - $\partial \mathrm{VR}(p, S) \not\subset \mathrm{VR}(p, S)$
  - $V(S)$ is the union of all the Voronoi edges
- Voronoi Edge $e$ between $\mathrm{VR}(p, S)$ and $\mathrm{VR}(q, S)$

$$e = \partial \mathrm{VR}(p, S) \cap \partial \mathrm{VR}(q, S)$$

- Voronoi Vertex $v$ among $\mathrm{VR}(p, S)$, $\mathrm{VR}(q, S)$, and $\mathrm{VR}(r, S)$

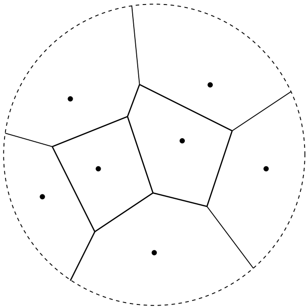
$$v = \partial \mathrm{VR}(p, S) \cap \partial \mathrm{VR}(q, S) \cap \partial \mathrm{VR}(r, S)$$

# Complexity of $V(S)$

**Theorem**

$V(S)$ has $O(n)$ edges and vertices. The average number of edges of a Voronoi region is less than $6$.

- Add a large curve $\Gamma$
  - $\Gamma$ only passes through unbounded edges of $V(S)$
  - Cut unbounded pieces outside $\Gamma$
  - One additional face and several edges and vertices.



# Complexity of $V(S)$

**Theorem**

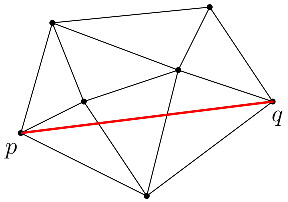
$V(S)$ has $O(n)$ edges and vertices. The average number of edges of a Voronoi region is less than $6$.

- Euler's Polyhedron Formula: $v - e + f = 1 + c$
  - $v$: # of vertices, $e$: # of edges, $f$: # of faces, and $c$: # number of connected components.
- An edge has two endpoints, and a vertex is incident to at least three edges.
  - $3v \leq 2e \to v \leq 2e/3$
- $f = n + 1$ and $c = 1$
  - $v = 1 + c + e - f = e + 1 - n \leq 2e/3 \to e \leq 3n - 3$
  - $e = v + f - 1 - c = v + n - 1 \geq 3v/2 \to v \leq 2n - 2$
- Average number of edges of a region $\leq (6n - 6)/n < 6$

# Triangulation

**Definition**

Given a set $S$ of points on the plane, a triangulation is maximal collection of non-crossing line segments among $S$.




Crossing ($\overline{pq}$)

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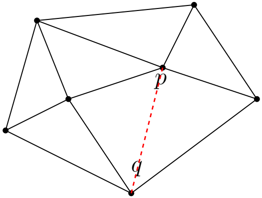


Not Maximal ($\overline{pq}$ is allowable)

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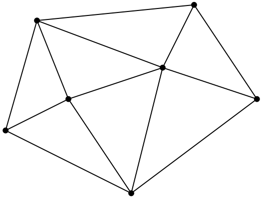
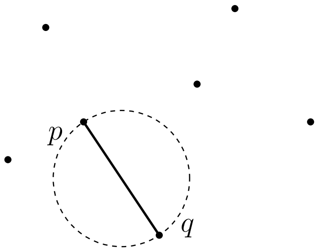


**Triangulation**

# Delaunay Edge

**Definition**

An edge $\overline{pq}$ is called **Delaunay** if there exists a circle passing through $p$ and $q$ and containing no other point in its interior.

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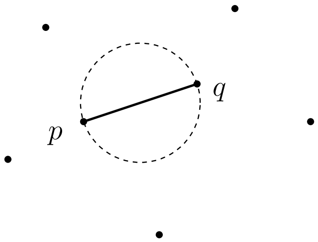


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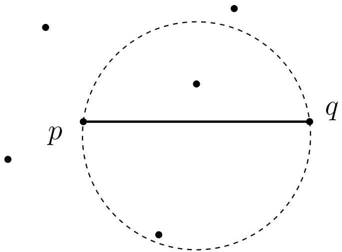
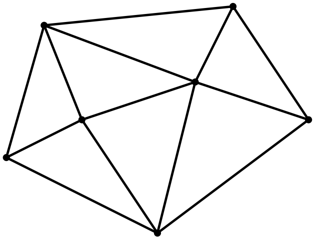



$\overline{pq}$ is **NOT** Delaunay

# Delaunay Triangulation

**Definition**

A **Delaunay Triangulation** is a triangulation whose edges are all Delaunay.



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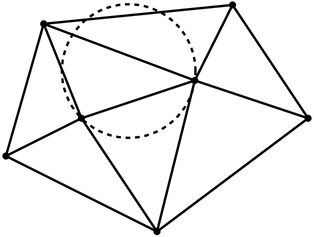

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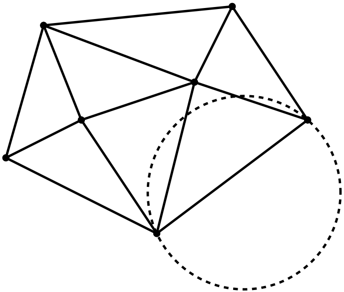

# Delaunay Triangulation

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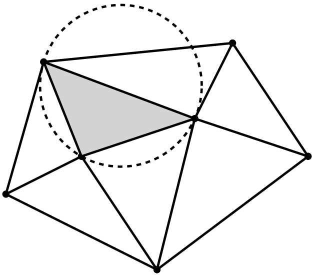
- For each face, there exists a circle passing all its vertices and containing no other point.



# General Position Assumption

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   - Each face of the Delaunay triangulation is a **triangle**.

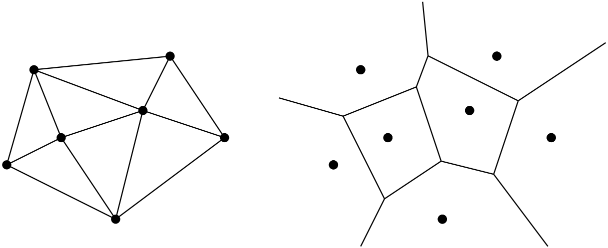
- There is a unique Delaunay triangulation.

# Duality

**Theorem**

Under the general position assumption, the Delaunay triangulation is a dual graph of the Voronoi diagram.

- A site $p \leftrightarrow$ a Voronoi region $\mathrm{VR}(p, S)$



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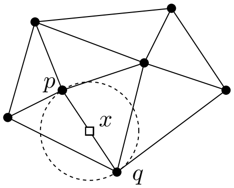


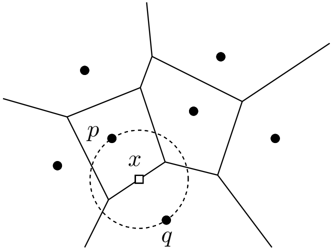

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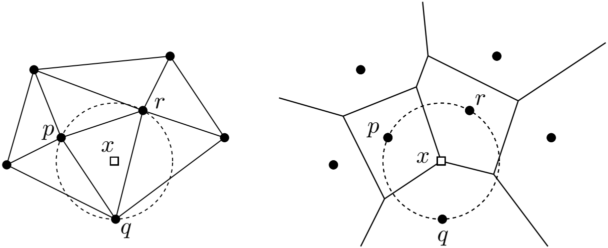

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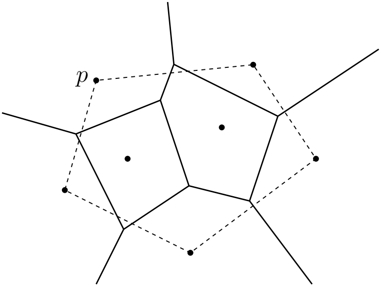

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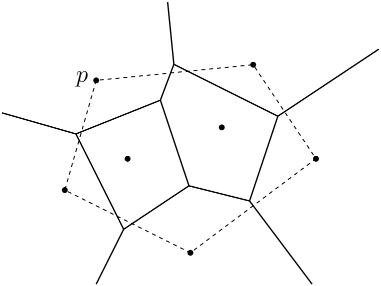


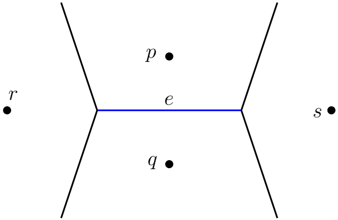

- $O(n \log n)$ time algorithms
  - Plane Sweep Algorithm
  - Divide and Conquer Algorithm

# Randomized Incremental Construction

- General Idea
  - Consider a random sequence of $S$, $(s_1, s_2, \ldots, s_n)$.
  - Let $R_i$ be $\{s_1, \ldots, s_i\}$
  - From $i = 4$ to $i = n - 1$, construct $V(R_{i+1})$ from $V(R_i)$ by inserting $s_{i+1}$.
- Tasks
  - What is a configuration?
  - What is a conflict relation?
  - How to use conflict relations to insert a site?
  - How to update conflict relations?
- General Position Assumption
  - No more than three sites are located on the same circle $\rightarrow$ The degree of a Voronoi vertex is exactly 3
  - No more than two points are located on the same line $\rightarrow$ The Voronoi diagram is connected

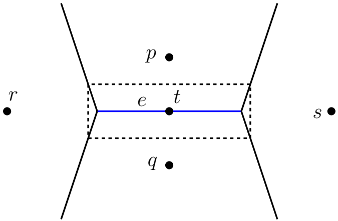
# Configuration: A Voronoi edge

- A Voronoi region can not be a configuration because it could consist of $O(n)$ edges, i.e., it is not defined by a constant number of sites
- Consider a Voronoi edge $e$ between $\mathrm{VR}(p, S)$ and $\mathrm{VR}(q, S)$
  - $e \subseteq B(p, q)$
  - Assume $e$ has two endpoints $v$ and $u$. Then $v = \overline{\mathrm{VR}(p, S)} \cap \overline{\mathrm{VR}(q, S)\mathrm{VR}(r, S)}$ and $u = \mathrm{VR}(p, S) \cap \mathrm{VR}(q, S)\mathrm{VR}(s, S)$.
  - $e$ is defined by $p$, $q$, $r$, $s$
  - A Voronoi edge is defined by at most 4 sites.



# Conflict Relation

- A site $t \in S \setminus R$ conflicts with a Voronoi edge $e$ between $\mathrm{VR}(p, R)$ and $\mathrm{VR}(q, R)$ if $e \cap \mathrm{VR}(t, R \cup \{t\}) \neq \emptyset$.



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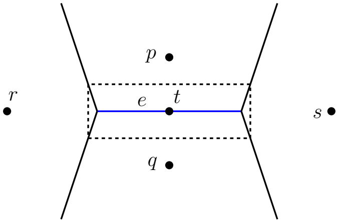


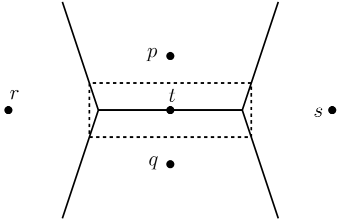

**Lemma**

$e \cap \mathrm{VR}(r, R \cup \{r\}) = e \cap \mathrm{VR}(r, \{p, q, r\})$ (Local Test)

# Insert a Site $t$

**Lemma**

$V(R) \cap \mathrm{VR}(t, R \cup \{t\})$ is a tree



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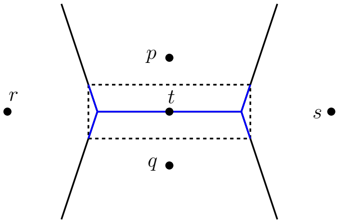

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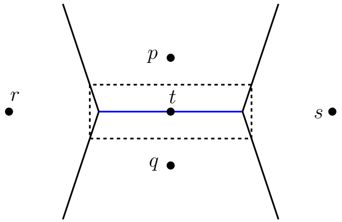


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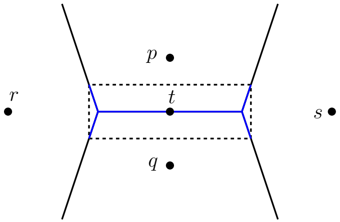


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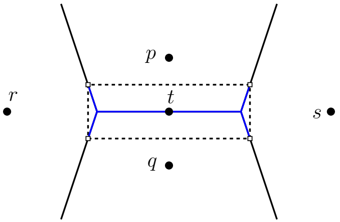


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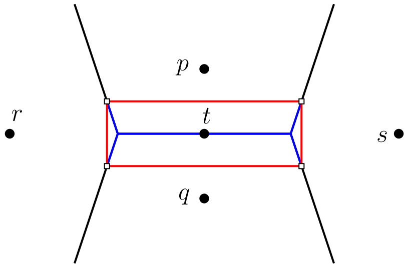


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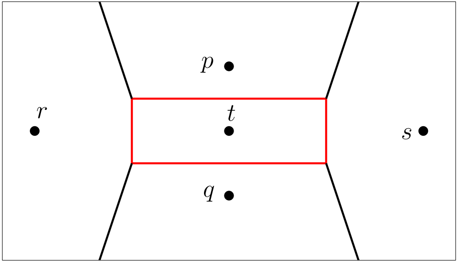


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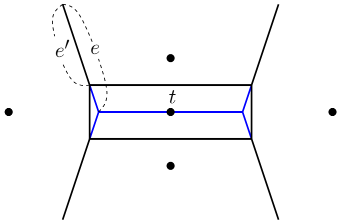
# Update Conflict Relations: Partial Edges

- Consider an edge $e'$ of $V(R \cup \{t\})$ which belongs to an edge $e$ of $V(R)$



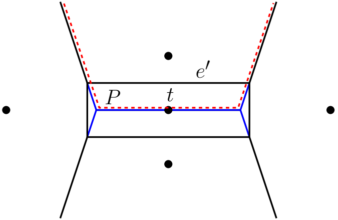

**Lemma**

Any site $s \in S \setminus (R \cup \{t\})$ in conflict with $e'$ will conflict with $e$. That is, if $e' \cap \mathrm{VR}(s, R \cup \{t, s\}) \neq \emptyset$, $e \cap \mathrm{VR}(s, R \cup \{s\}) \neq \emptyset$.

- The set of sites in conflict with $e'$ is a subset of the set of sites in conflict with $e$
- For each site in conflict with $e$, check if it conflicts with $e'$.

# Update Conflict Relations: Fully new edges

- Consider an edge $e'$ of $V(R \cup \{t\})$ which does not belong to any edge of $V(R)$




**Lemma**

$e'$ and a path of $V(R) \cap \mathrm{VR}(t, R \cup \{t\})$ will form a cycle. Let $P$ be the path in $V(R) \cap \mathrm{VR}(t, R \cup \{t\})$ which forms a cycle with $e'$. Any site $s \in S \setminus (R \cup \{t\})$ in conflict with $e'$ will conflict with one edge along the path.

- For each site in conflict with an edge of $P$, check if it conflicts with $e'$.

# The number of updates

**Lemma**

Each edge of $V(R)$ which is destroyed due to the insertion of $t$ will be check at most 3 times.

- An edge of $V(R)$ contains at most one edge $V(R \cup \{t\})$ and belongs to at most two paths which form a cycle with an edge of $V(R \cup \{t\})$ .

**Lemma**

The time to insert $t$ is proportional to the total size of the conflict lists for the edges of $V(R)$ which are destroyed due to the insertion of $t$

Thank You!!