Appl. of WSPD $\qquad S$ = set of $n$ points in $\mathbb{R}^d$

Thm
Closest Pair in $S$: $\quad O(n \log n) \quad$ Build WSPD w.r.t. $s > 2$
check all singleton pairs

Thm (without proof) $\quad O(n \log n + nk) \quad$ Needs WSPD + auxiliary structure
$k$ nearest neighbors to each point in $S$

$k=1$: $O(n \log n)$

so far: as good as VD in $d=2$

Post office $\quad$ Given $q \in \mathbb{R}^d$ arbitrary, find closest $p \in S$

(Curse of dimensionality)

with VD in $\mathbb{R}^2$: very efficient solutions known:
preprocess VD + point location structure of size $O(n)$ in time $O(n \log n)$
$O(\log n)$ per query

nothing like this known in $d \gg 2$

Only approx. solutions known: given $\varepsilon$, $q$, report $p' \in S$ such that
$|qp'| \leq (1+\varepsilon)\,|qp|$, $\quad p$ = real NN of $q$ in $S$

Use, e.g.: dynamic $(1+\varepsilon)$-spanner $G$ of $S$.
insert $q$ in $G \longrightarrow G'$
determine $p'$ := nearest of all direct neighbors of $q$ in $G'$
report $p'$

Let $p :=$ nearest neighbor of $q$ in $S$
$\pi :=$ shortest path in $G'$ from $q$ to $p$

$\Rightarrow$ first edge of $\pi$ has length $\geq |qp'|$

$\Rightarrow |qp'| \leq |\pi| \leq (1+\varepsilon)|qp|$, since $G'$ $(1+\varepsilon)$-spanner.

Can be implemented in $O(n \log n)$ preproc, $O(n)$ size
$O(\log n)$ query time.

(just as VD).

<u>More apps of WSPD:</u>

<u>Thm</u> $(1+\varepsilon)$-Spanner in time $O(n \log n)$ with $O(n)$ edges

build WSPD
pick one edge
per pair.

<u>Apps of Spanners</u>

① Road networks

② Approx of MST

in $G=(V,E)$ in time $O(|E| \log |E|)$
Kruskal

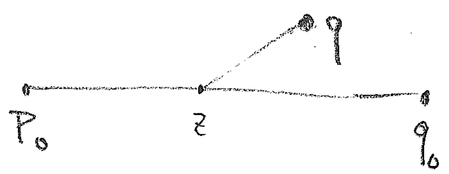
For points in $\mathbb{R}^d$: $G =$ complete graph
$|E| = \binom{n}{2} \in \Theta(n^2)$

Faster MST in $d=2$ : 

Lemma 1 $S = P \dot\cup Q$, $\quad |p_0 q_0| = \min\{|pq| \mid p \in P, q \in Q\}$

$\Rightarrow$ $\overline{p_0 q_0}$ Delaunay edge in $D(S)$ ($p_0, q_0$ neighboring Voronoi regions)

Proof



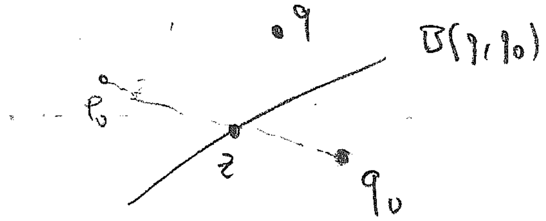

let $z \in \overline{p_0 q_0}$, assume $z \in \overline{VR}(q, S)$, $q \in Q \setminus \{q_0\}$

$\Rightarrow$ $|p_0 q| \le |p_0 z| + |zq| \le |p_0 z| + |z q_0| = |p_0 q_0| \underset{\uparrow \text{ min}}{\le} |p_0 q|$

$\Rightarrow$ $=$ everywhere

$\Rightarrow$ $|zq| = |zq_0|$ and $|p_0 q_0| = |p_0 q|$

$\Rightarrow$




from sketch: $|p_0 q| < |p_0 q_0|$ ↯.

□

Lemma 2 Each edge of MST(S) is a Delaunay edge

Proof Lemma 1 + Kruskal

Thm MST in $O(n \log n)$

Proof Construct $D(S)$. Run Kruskal on these edges only.

In $d > 2$? $V(S)$, $D(S)$ don't work efficiently.

But



**Thm** Can construct Tree $T$ over $S$ in time $O(n \log n)$
s.th. $|T| \le (1+\varepsilon)\,|MST|$

**Proof** $G := (1+\varepsilon)$-spanner of $S$ ($O(n \log n)$ time, $O(n)$ size)
$T := MST(G)$ ($O(n \log n)$ Kruskal)

Let $e_i := (p_i, q_i)$ edge of "real" $MST(S)$

$\Rightarrow$ ex shortest path $\pi_i$ from $p_i$ to $q_i$ in $G$, $|\pi_i| \le (1+\varepsilon)\,|p_i q_i|$

Let $M := \bigcup \pi_i$ $\Rightarrow$ $M$ is a spanning graph of $G$

$$\Rightarrow \quad |T| \le |M| \le \sum_i |\pi_i| \le \sum_i (1+\varepsilon)\,|p_i q_i| = (1+\varepsilon)\,|MST(S)|$$

$\square$

MST useful because of minimum weight (= sum of edge lengths), not because of low dilation.

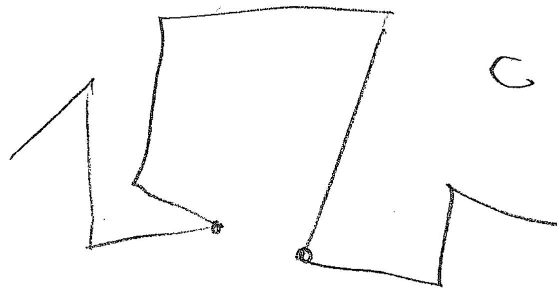
But: with $k$ extra edges, carefully placed,
dilation of $O\left(\frac{n}{k+1}\right)$ can be obtained, in time $O(n \log n)$
(Aronov et al '06)

Optimization is NP-hard (Klein, Kutz '06)

## One more Spanner app

dilation of a chain

trivial (?) $O(n)$




Thm Can, in time $O(n \log n)$, compute $\delta'$ such that

$$\delta(C) \underset{(i)}{\le} \delta' \underset{(ii)}{\le} (1+\varepsilon)\,\delta(C)$$

Proof $G := (1+\varepsilon)$-Spanner of $C$ $\quad O(n \log n)$

For each edge $(p,q)$ of $G$: compute $\delta(p,q)$ $\quad O(n)$

report $\delta' :=$ max of these values

clear: $\delta(C) \le \delta'$ (i)

Proof of (ii): Assume $\delta(C) = \delta(p,q)$, $p,q$ vertices of $C$

$\pi :=$ Shortest path from $p$ to $q$ in $G$ : $|\pi| \le (1+\varepsilon)|pq|$

$\pi = e_1 \dots e_r$, $e_i = (q_i, q_{i+1})$

$C_{q_i}^{q_{i+1}}$ := Segment of chain $C$ from $q_i$ to $q_{i+1}$

$$\Rightarrow \delta(C) = \delta(p,q) = \frac{|C_p^q|}{|pq|} \le \frac{\sum_i |C_{q_i}^{q_{i+1}}|}{|pq|} \le \frac{\sum_i |C_{q_i}^{q_{i+1}}|}{\frac{\pi}{(1+\varepsilon)}}$$

$$\le (1+\varepsilon)\frac{\sum_i |C_{q_i}^{q_{i+1}}|}{\sum_i |e_i|} \underset{\text{Trick}}{\le} (1+\varepsilon) \max_i \frac{|C_{q_i}^{q_{i+1}}|}{|e_i|} = (1+\varepsilon)\max_i \delta(q_i, q_{i+1})$$

$$\le (1+\varepsilon)\,\delta'$$

$\square$