Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Geometric Firefighting – Lower Bound and FF Curve

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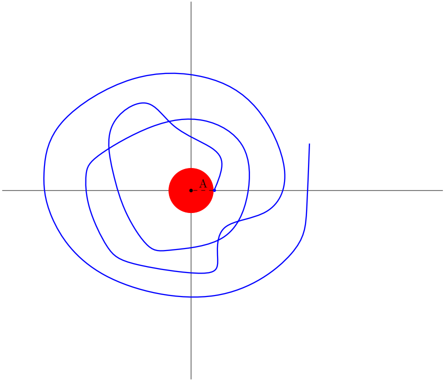
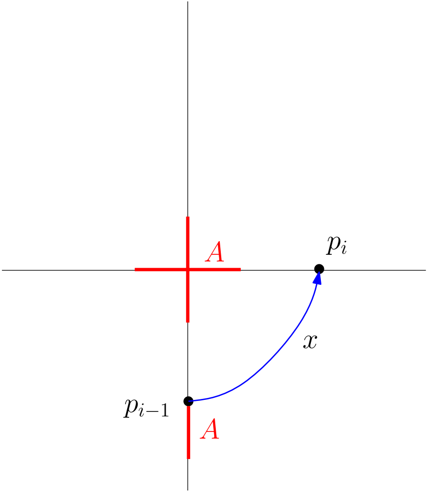
- Start at the fire!
- Spiralling strategies!
- Visit four axes in cyclic order
- Visit axes in increasing distance



Theorem 58: Each "spiralling" strategy must have speed $v > 1.618\ldots$ (golden ratio) to be successful.

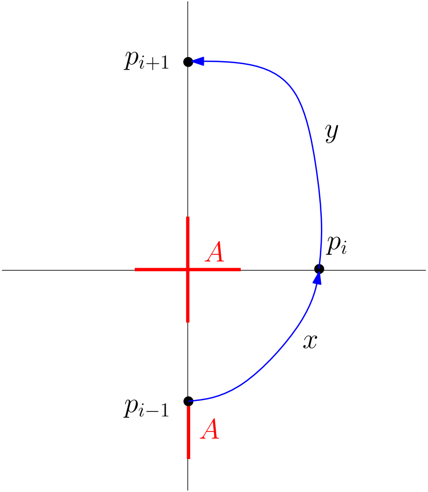
Proof of lower speed bound: suppose $v \leq 1.618$



By induction:
On reaching p_i,
interval of length A below
p_{i-1} is on fire.

(Induction base!)

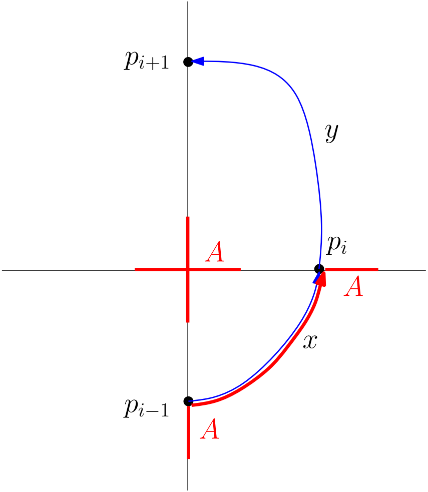
Proof of lower speed bound: suppose $v \leq 1.618$



Inductive Step:

After arriving p_{i+1}
fire moves at least $x + A$

Proof of lower speed bound: suppose $v \leq 1.618$



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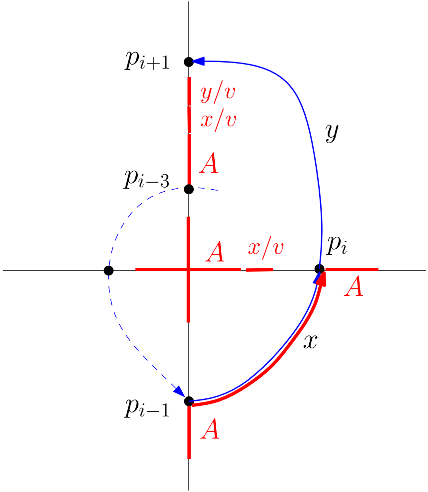


On reaching p_{i+1}:

1. $A + \frac{x}{v} \leq p_i \leq x$ and
2. $A + \frac{x}{v} + \frac{y}{v} \leq p_{i+1} \leq y$

$\Longrightarrow \frac{1}{v(v-1)}x + \frac{1}{v-1}A \leq \frac{y}{v}$

$\Longrightarrow x + A \leq \frac{y}{v}$

from $v^2 - v \leq 1$

FollowFire Strategy for $v = 5.27$!

Logarithmic spiral of excentricity α around Z ($\frac{1}{v} = \cos(\alpha)$)!

(First Part)

FollowFire Strategy for $v = 5.27$!

Logarithmic spiral of excentricity α around p_0 ($\frac{1}{v} = \cos(\alpha)$)!

(Second Part)

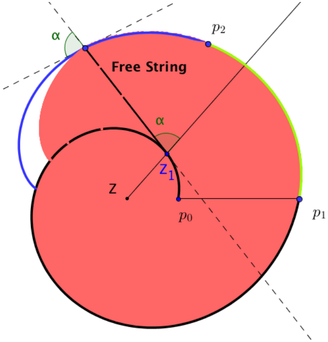
FollowFire Strategy for $v = 5.27$!

Excentricity α around wrapping center Z_1 ($\frac{1}{v} = \cos(\alpha)$)!

(Third part!)

FollowFire: Free String Wrapping!

- $v = 5.27$ $(\alpha = 1.38)$
- $\text{Log}(\mathbf{p_0}, \mathbf{p_1})$, $\text{Log}(\mathbf{p_1}, \mathbf{p_2})$
- Free string: $F_1(l)$:
  Wrapping around $\text{Log}(\mathbf{p_0}, \mathbf{p_1})$



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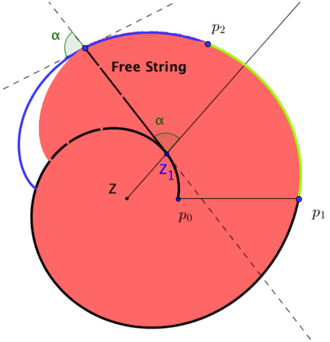
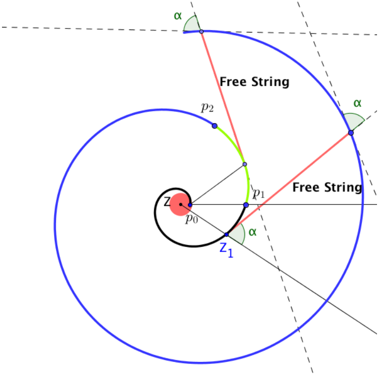


- $v = 3.07$ ($\alpha = 1.24$)
- Wrapping around $\mathrm{Log}(\mathbf{p_1}, \mathbf{p_2})$



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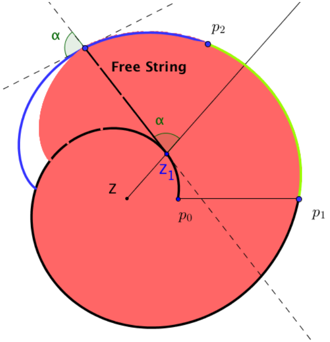
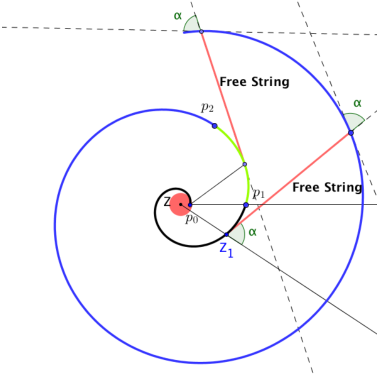


- $v = 3.07$ $(\alpha = 1.24)$
- Wrapping around $\text{Log}(\mathbf{p_1}, \mathbf{p_2})$

Wrapping around wrappings!



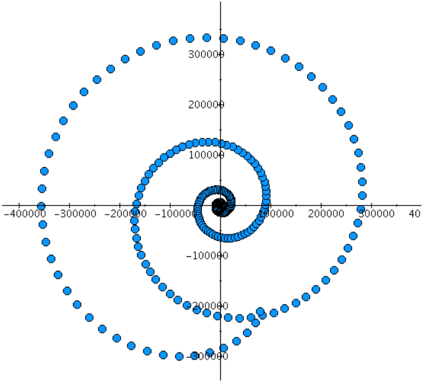
Experimental approach!

(Spiral Generator Appet!)

FollowFire: Successful?

$v = 2.69$ ($\alpha = 1.19$):
8 rounds!

$v = 2.593$ ($\alpha = 1.175$):
Simulation did not succeed!



Successful for which $v \in (1, \infty)$?

Lower and upper bounds on v! Proofs!

Upper bound by FollowFire

Theorem 59: FollowFire strategy is successful if $v > v_c \approx 2.6144$

Upper bound by FollowFire

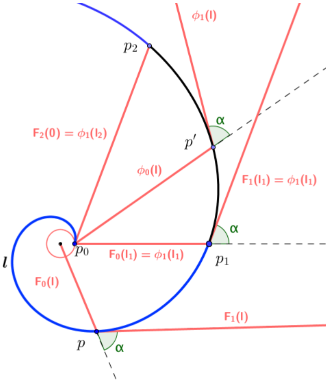
Theorem 59: FollowFire strategy is successful if $v > v_c \approx 2.6144$

Sketch! When gets the free string to zero?

1. Parameterize free strings for coil j (Linkage)
2. Structural properties
3. Successive interacting differential equations
4. Inserting end of parameter interval
5. Coefficients of power series
6. Ph. Flajolet: Singularities
7. Pringsheim's Theorem and Cauchy's Residue Theorem

FollowFire Wrapping process!

Free strings F_j/ϕ_j parameterized by lenght of starting spirals!



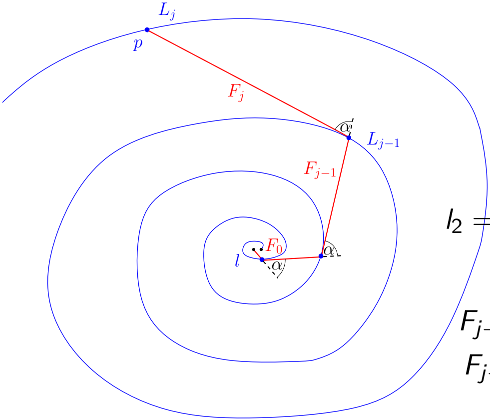
$$|Log(p_0, p_1)| = l_1$$
$$|Log(p_0, p_1)| + |Log(p_1, p_2)| = l_2$$

F_j: $l \in [0, l_1]$
ϕ_j: $l \in [l_1, l_2]$

Upper bound: 1. Parameterize the free string (Linkage)

FollowFire Drawing backwards tagents!

Free strings F_j/ϕ_j parameterized by lenght of starting spirals!



F_j: $l \in [0, l_1]$
ϕ_j: $l \in [l_1, l_2]$

$$l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$$

$$l_2 = \frac{A}{\cos \alpha}(e^{2\pi \cot \alpha} - 1)e^{\alpha \cot \alpha}$$

$$F_{j+1}(l_1) = \phi_{j+1}(l_1)$$
$$F_{j+1}(0) = \phi_j(l_2)$$

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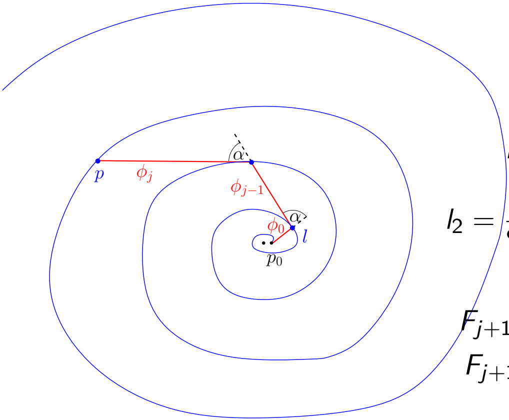

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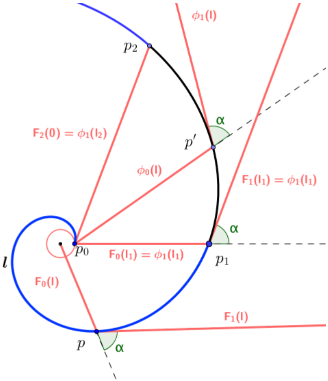


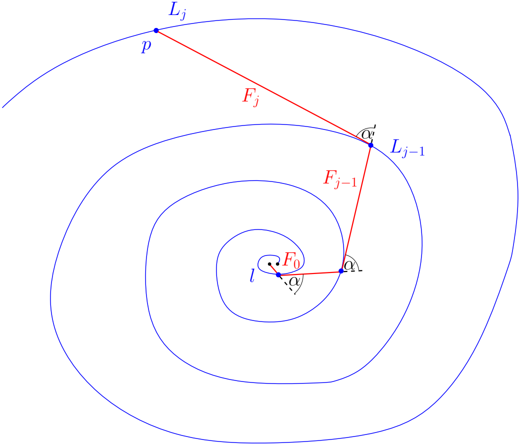
$$|Log(p_0, p_1)| = l_1$$
$$|Log(p_0, p_1)| + |Log(p_1, p_2)| = l_2$$

$$F_j\colon\ l \in [0, l_1]$$
$$\phi_j\colon\ l \in [l_1, l_2]$$

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \\ F_0(l) &= A + \cos(\alpha)\, l \end{aligned}$$

Parameterized by lenght l of starting spirals!

$L_j(l)$ length of the curve! $F_j(l)$ (and $\phi_j(l)$) length of the free string!



Lemma 60:

$$L_{j-1} + F_j = \cos\alpha \, L_j$$

Lemma 61:

$$\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$$

Helping Lemmata

Lemma 60: $L_{j-1} + F_j = \cos\alpha \, L_j$

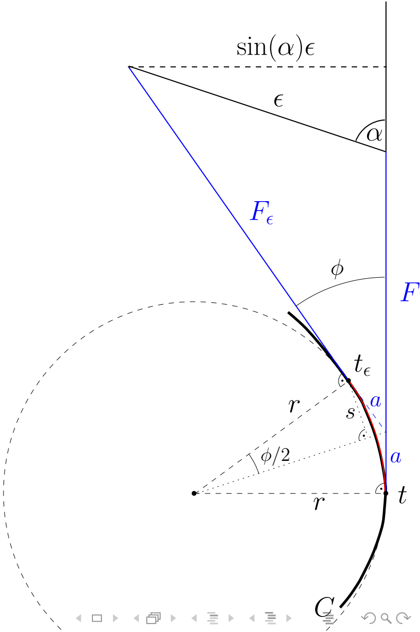
- Fire and fire fighter, reach endpoint at $F_j(l)$ at the same time
- Unit-speed fire, geodesic distance of $L_{j-1}(l) + F_j(l)$
- Fighter distance of $L_j(l)$ at speed $1/\cos\alpha$

Lemma 61: $\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$

Lemma 62: String of length F is tangent to point t on smooth curve C. End of string moves distance ϵ in direction α. For the curve length $C_t^{t_\epsilon}$ between t and the new tangent point, t_ϵ, we have

$$\lim_{\epsilon \to 0} \frac{C_t^{t_\epsilon}}{\epsilon} = \frac{r \sin \alpha}{F}$$

where r denotes radius of osculating circle at t.



Helping Lemmata

$r\,\sin(\phi/2) = s = a\,\cos(\phi/2)$
gives $a = r\,\tan(\phi/2)$

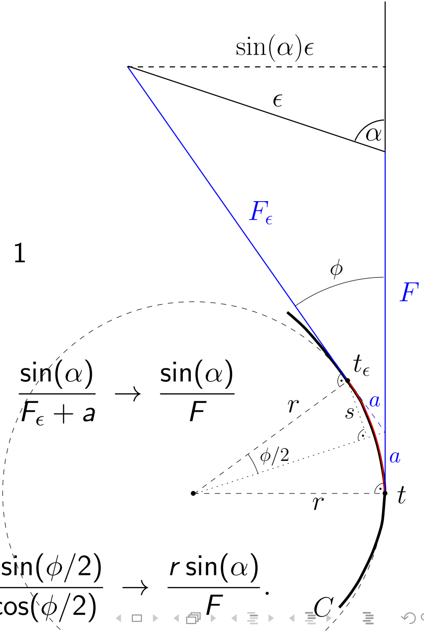
$2a$ approximates $c := C_t^{t_\epsilon}$:

$$\frac{c}{2a} = \frac{r\,\phi}{2r\,\tan(\phi/2)} \approx \cos^2(\phi/2) \rightarrow 1$$

$$\frac{\epsilon\,\sin(\alpha)}{\sin(\phi)} = \frac{F_\epsilon + a}{\sin(\pi/2)} \text{ gives } \frac{\sin(\phi)}{\epsilon} = \frac{\sin(\alpha)}{F_\epsilon + a} \rightarrow \frac{\sin(\alpha)}{F}$$

$\sin(\phi/2)/\epsilon \rightarrow \sin(\alpha)/(2F)$

$$\frac{C_t^{t_\epsilon}}{\epsilon} = \frac{c}{2a}\,\frac{2a}{\epsilon} \approx \frac{2r\,\tan(\phi/2)}{\epsilon} = \frac{2r\,\sin(\phi/2)}{\epsilon\,\cos(\phi/2)} \rightarrow \frac{r\,\sin(\alpha)}{F}.$$



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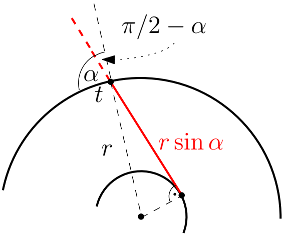

Lemma 63: Let t be point on smooth curve C, osculating circle at t has radius r. Lines L_s resulting from turning the normal at points s by angle of $\pi/2 - \alpha$. Limit intersection point of normals with L_t has distance $\sin \alpha \, r$ to t.

Helping Lemmata

Lemma 61: $\frac{L_j'}{L_{j-1}'} = \frac{F_j}{F_{j-1}}$

- Curve L_{j-1}, F_j, F_{j-1} depending on L_j, which depends on l
- Lemma 62: $\frac{L_{j-1}'(L_j)}{L_j'(L_j)} = L_{j-1}'(L_j) = \frac{r \sin\alpha}{F_j(L_j)}$
- FF Curve: Normal turned by $\pi/2 - \alpha$, tangents to previous coil
- Lemma 63: $F_{j-1}(L_j) = r \sin\alpha$
- Substitute L_j with $L_j(l)$ (derivatives cancel out!):

$$\frac{L_j'}{L_{j-1}'} = \frac{F_j}{F_{j-1}}$$

Build and solve differential equations

1. $\frac{F_j(l)}{F_0(l)} = \frac{L_j'(l)}{l'} = L_j'(l)$ (Lemma 61/Multiplication)

2. $F_j'(l) + L_{j-1}'(l) = \cos\alpha \, L_j'(l)$ (Lemma 60/Derivatives)

1+2 = Lin. Diff. Eq. $F_j'(l) - \frac{\cos(\alpha)}{F_0(l)} F_j(l) = -\frac{F_{j-1}(l)}{F_0(l)}$.

Textbook solution of $y'(x) + f(x)y(x) = g(x)$

$$y(x) = \exp(-a(x)) \left(\int g(t) \exp(a(t)) \, \mathrm{d}t + \kappa \right)$$

With $a = \int f$ and constant κ

Build and solve differential equations

$$F_j'(l) \;-\; \frac{\cos(\alpha)}{F_0(l)}\, F_j(l) \;=\; -\,\frac{F_{j-1}(l)}{F_0(l)}\,.$$

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$$a(l) \;=\; \int -\frac{\cos(\alpha)}{A + \cos(\alpha)\, l} \;=\; -\ln(F_0(l))$$

because of $F_0(l) = A + \cos(\alpha)\, l$, and we obtain

$$F_j(l) \;=\; F_0(l)\Big(\kappa_j \;-\; \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big).$$

Build and solve differential equations

For $l \in [0, l_1]$, F-linkages:

$$F_j(l) \;=\; F_0(l)\Big(\kappa_j \;-\; \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big). \qquad (1)$$

Same arguments and $l \in [l_1, l_2]$, ϕ-linkages:

$$\phi_j(l) \;=\; \phi_0(l)\Big(\lambda_j \;-\; \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)}\,\mathrm{d}t\Big). \qquad (2)$$

Successively resolve the constants κ_j, λ_j by:

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \end{aligned}$$

Build and solve differential equations

$$F_j(l) \;=\; F_0(l)\Big(\kappa_j \;-\; \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big). \tag{3}$$

$$\phi_j(l) \;=\; \phi_0(l)\Big(\lambda_j \;-\; \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)}\,\mathrm{d}t\Big). \tag{4}$$

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \end{aligned}$$

$F_{-1} = \phi_{-1} = 0,\; F_0(l) = A + l\cos\alpha,\; \kappa_0 = 1,\; \lambda_0 = 1$

Example: κ_1 with $\phi_0(l_2) = F_1(0)$ gives

$$\kappa_1 := \frac{\phi_0(l_2)}{F_0(0)} \;+\; \int \frac{F_0(t)}{F_0^2(t)}\mathrm{d}t|_{l=0}$$

Build and solve differential equations

In general:

$$\kappa_{j+1} := \frac{\phi_j(l_2)}{F_0(0)} + \int \frac{F_j(t)}{F_0^2(t)}\mathrm{d}t|_{l=0}$$

$$\lambda_{j+1} := \frac{F_{j+1}(l_1)}{\phi_0(l_1)} + \int \frac{\phi_j(t)}{\phi_0^2(t)}\mathrm{d}t|_{l=l_1}$$

so that

$$\begin{aligned} F_{j+1}(l) &= F_0(l)\left(\frac{\phi_j(l_2)}{F_0(0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)}\,\mathrm{d}t\right), \\ \phi_{j+1}(l) &= \phi_0(l)\left(\frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)}\,\mathrm{d}t\right), \end{aligned}$$

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \end{aligned}$$

Final formulas:

$$F_{j+1}(l) \;=\; F_0(l)\left(\frac{\phi_j(l_2)}{F_0(0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)}\,\mathrm{d}t\right),$$

$$\phi_{j+1}(l) \;=\; \phi_0(l)\left(\frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)}\,\mathrm{d}t\right),$$

For simplicity, let us write

$$G_j(l) := \frac{F_j(l)}{F_0(l)} \quad \text{and} \quad \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)},$$

which leads to

$$G_{j+1}(l) \;=\; \frac{\phi_0(l_2)}{F_0(0)}\,\chi_j(l_2) - \int_0^l \frac{G_j(t)}{F_0(t)}\,\mathrm{d}t$$

$$\chi_{j+1}(l) \;=\; \frac{F_0(l_1)}{\phi_0(l_1)}\,G_{j+1}(l_1) - \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)}\,\mathrm{d}t.$$

Final formulas:

$$G_j(l) := \frac{F_j(l)}{F_0(l)} \text{ and } \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)},$$

which leads to

$$\begin{aligned} G_{j+1}(l) &= \frac{\phi_0(l_2)}{F_0(0)}\,\chi_j(l_2) - \int_0^l \frac{G_j(t)}{F_0(t)}\,\mathrm{d}t \\ \chi_{j+1}(l) &= \frac{F_0(l_1)}{\phi_0(l_1)}\,G_{j+1}(l_1) - \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)}\,\mathrm{d}t. \end{aligned}$$

Lemma 64: The curve encloses the fire if and only if there exists an index j such that $F_j(l_1) \leq 0$ holds.