Properties of Abstract Voronoi Diagrams

A simplified version of the following reference

- Rolf Klein, "Combinatorial Properties of Abstract Voronoi Diagrams," Graph-Theoretic Concepts in Computer Science (WG 89), LNCS 834, pp. 356–369, 1989.

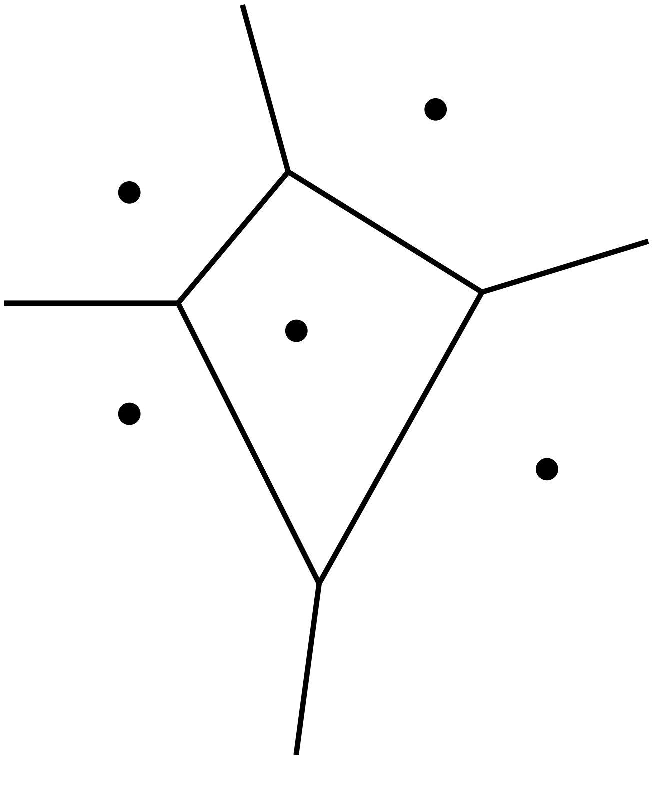
Euclidean Voronoi Diagrams

Voronoi Diagram: Given a set S of n point sites in the plane, the Voronoi diagram $V(S)$ of S is a planar subdivision such that

- Each site $p \in S$ is assigned a Voronoi region denoted by $\mathrm{VR}(p, S)$
- All points in $\mathrm{VR}(p, S)$ share the same nearest site p in S

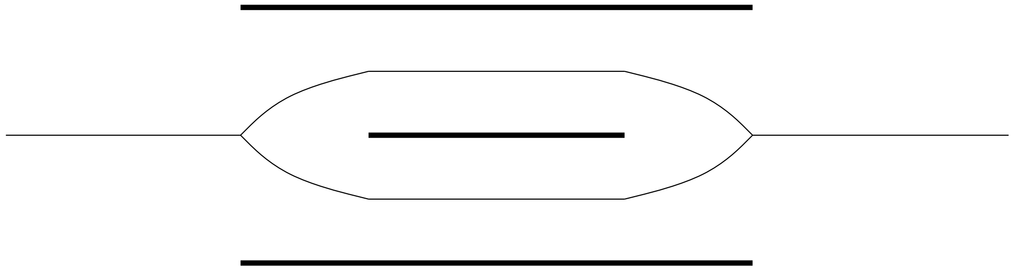
Voronoi Edge: The common boundary between two adjacent Voronoi regions, $\mathrm{VR}(p, S)$ and $\mathrm{VR}(q, S)$, i.e., $\mathrm{VR}(p, S) \cap \mathrm{VR}(q, S)$,is called a *Voronoi edge*.

Voronoi Vertex: The common vertex among more than two Voronoi regions is called a *Voronoi vertex*.

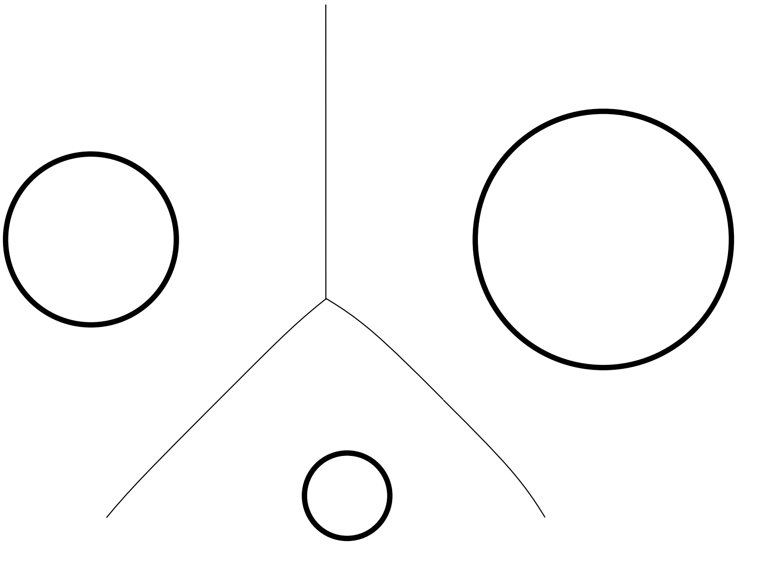


The Euclidean Voronoi diagram can be computed in $O(n \log n)$ time

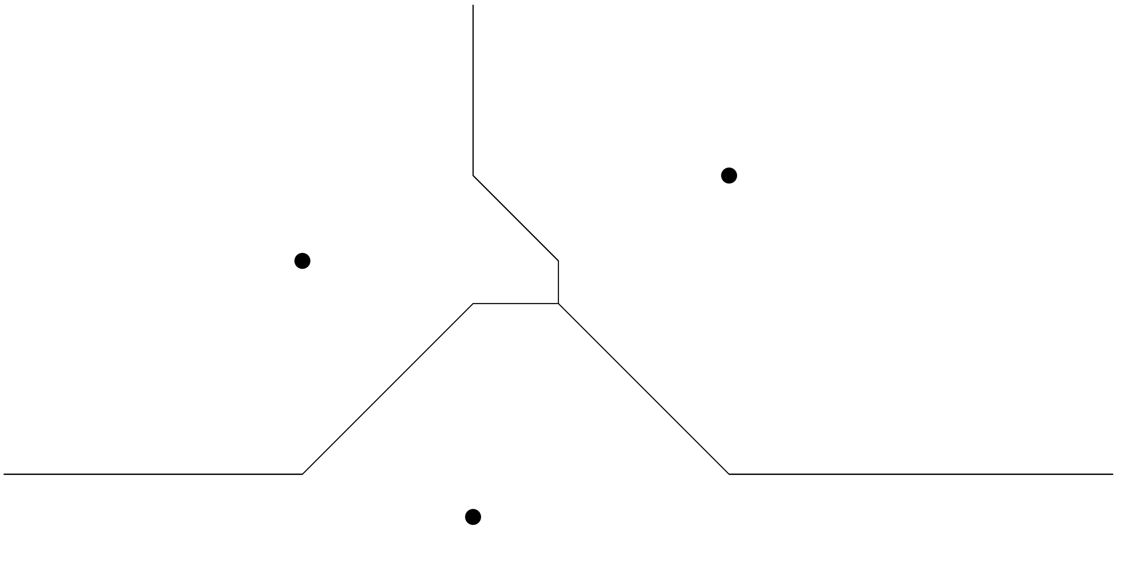
- Line Segment Voronoi Diagram



- Circle Voronoi Diagram



- Voronoi Diagram in the L_1 metric



Bisecting Systems

- For two sites $p, q \in S$, the bisector $\boldsymbol{J(p,q)}$ between p and q is defined as $\{x \in R^2 \mid d(x,p) = d(x,q)\}$

- $J(p,q)$ partitions the plane into two half-planes
  - $D(p,q) = \{x \in R^2 \mid d(x,p) < d(x,q)\}$
  - $D(q,p) = \{x \in R^2 \mid d(x,q) < d(x,p)\}$

- $\mathrm{VR}(p,S) = \bigcap_{q \in S \setminus \{p\}} D(p,q)$

- $V(p,S) = R^2 \setminus \bigcup_{p \in S} \mathrm{VR}(p,S)$
  - consists of Voronoi edges.

Abstract Voronoi Diagrams

A unifying approach to computing Voronoi diagrams among different geometric sites under different distance measures.

A bisecting system $\mathcal{J} = \{J(p,q) \mid p, q \in S\}$ for a set S of sites (indices)

A bisecting system $\mathcal{J}$ is **admissible** if $\mathcal{J}$ satisfies the following axioms

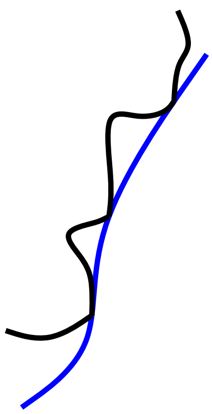
(A1) Each bisecting curve in $\mathcal{J}$ is homeomorphic to a line (not closed)

(A2) For each non-empty subset S' of S and for each $p \in S'$, $\mathrm{VR}(p,S')$ is path-connected.

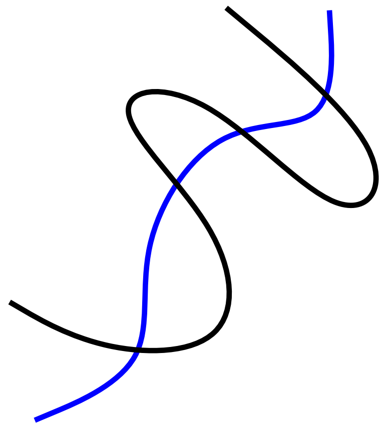
(A3) For each non-empty subset S', $R^2 = \bigcup_{p \in S'} \overline{\mathrm{VR}(p,S')}$

(A4) Any two curves in $\mathcal{J}$ have only finitely many intersection points, and these intersections are transversal.

- (A1) can be written as "Each curve in $\mathcal{J}$ is unbounded. After stereographic projection to the sphere, it can be completed to a closed Jordan curve through the north pole."
- (A4) can be removed through several complicated proofs.

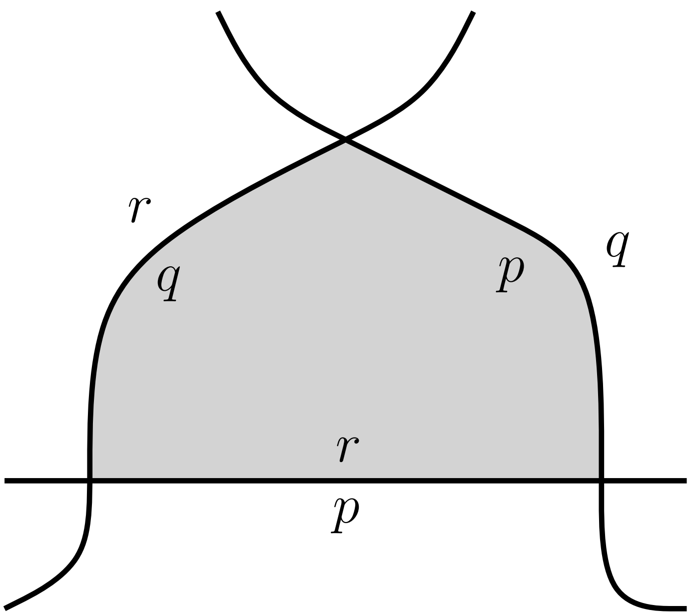


Not Transversal

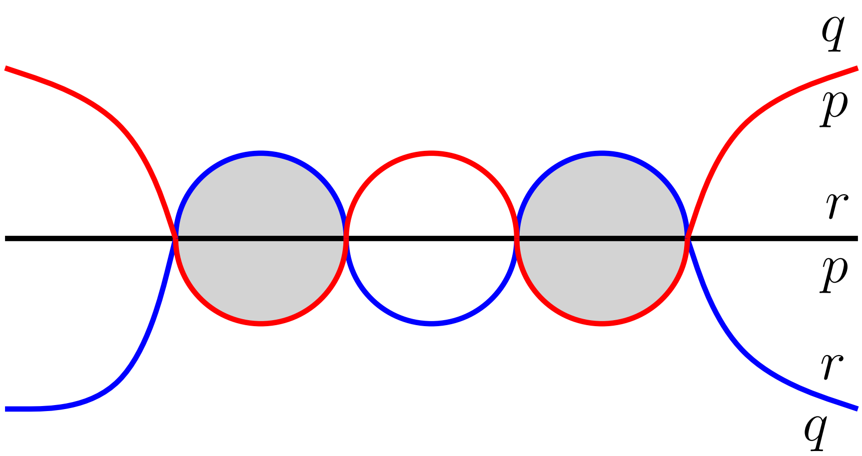


Transversal

Not Admissible

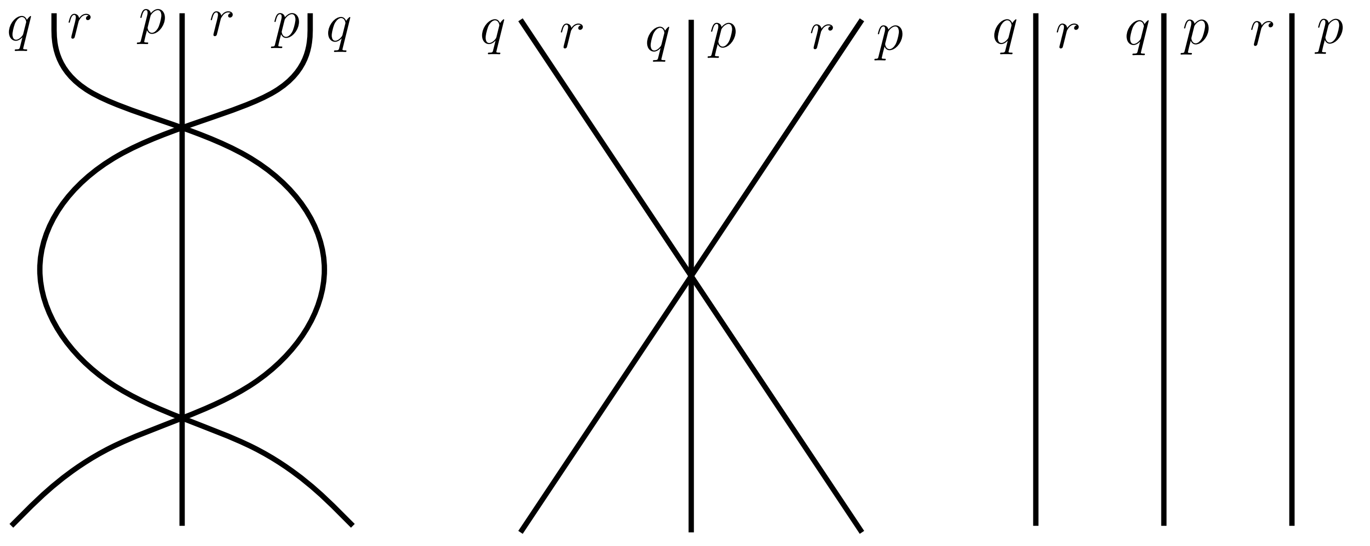


Non-Man Land



Disconnected

Three possibilities of an admissible system for three sites



Abstract Voronoi Diagrams

- A category of Voronoi diagrams
  - points in any convex distance function
  - Karlsruhe metric
  - Line segments and convex polygons of constant size

Basic Properties

Lemma 1
Let $(S, \mathcal{J})$ be a bisecting curve system. If for each nonempty subset $S' \subseteq S$, $R^2 = \bigcup_{p \in s'} \overline{\mathrm{VR}(p, S')}$ (Axiom (A3)), then
for every three pairwise sites p, q, and r in S, $D(p, q) \cap D(q, r) \subseteq D(p, r)$ (Transitivity).

Proof:

- Let x be a point in $D(p,q) \cap D(q,r)$.
- x must be contained in $D(r,p)$, $J(r,p)$, and $D(p,r)$.
- If x were contained in $D(r,p)$, it could not lie in the closure of any closed Voronoi region:

$$\overline{\mathrm{VR}(q,S')} \subseteq \overline{D(q,p)} = D(q,p) \cup J(q,p)$$
$$\overline{\mathrm{VR}(r,S')} \subseteq \overline{D(r,q)} = D(r,q) \cup J(r,q)$$
$$\overline{\mathrm{VR}(p,S')} \subseteq \overline{D(p,r)} = D(p,r) \cup J(p,r)$$

  for $S' = \{p,q,r\}$. This contraicts Axiom (A3).

- If $x \in J(p,r)$, consider a small neighborhood U at x such that $U \subseteq D(p,q) \cap D(q,r)$.
  - There is an arc α with endpint x such that $\alpha \subset U$ and $\alpha \setminus \{x\} \subset D(r,p)$.
  - Inside U, α contains a point $z \in D(p,q) \cap D(q,r) \cap D(r,p)$
  - z does not lie in any of $\mathrm{VR}(p,S')$, $\mathrm{VR}(q,S')$, and $\mathrm{VR}(r,S')$ for $S' = \{p,q,r\}$, contradicting Axiom (A3).
- To conclude, $x \in D(p,r)$.