## **Nmber of faces of convex polytope** (Chapter 5.4 and 5.5)

**Main Question**:
For a convex polytope $P$ with $n$ vertices in $\mathbb{R}^d$, what is the maximum number of facets or faces of $P$?

Terminology

- $f_j = f_j(P)$ denotes the number of $j$-faces of a polytope $P$
- The vector $(f_0, f_1, \ldots, f_d)$ is called the *$f$-vector* of $P$
- $f_0 = n$
- Estimate $f_{d-1}$ and $\sum_{k=0}^{d} f_k$.

A 3-dimensional polytope

- $f_1 \leq 3n - 6$ and $f_2 \leq 2n - 4$
  - By well-known results for planar graphs
- Equalities hold if and only if the polytope is simplicial (all facets are triangles)

Higher dimensions

- For every $n \geq 5$, a 4-dimensional convex polytope with $n$ vertices can have every two vertices connected by an edge, i.e., $\binom{n}{2}$ edges
- In any fixed dimension $d$, the number of facets can be of order $n^{\lfloor d/2 \rfloor}$

The number of faces for a given dimension and number of vertices is the largest possible for so-called *Cyclic Polytopes*

## Moment curve

The curve $\gamma = \{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\}$

## Cyclic polytope

The convex hull of finitely many points on the moment curve is called *a cyclic polytope.*

## Lemma

Any hyperplane $h$ intersects the moment curve $\gamma$ in at most $d$ points.
If there are $d$ intersections, then $h$ cannot be tangent to $\gamma$, and thus at each intersection, $\gamma$ passes from one side of $h$ to the other

Sketch of Proof

- $h$ can be expressed by the equation $\langle a, x \rangle = b$, or in coordinates $a_1x_1 + a_2x_2 + \cdots + a_dx_d - b = 0$.
- A point of $\gamma$ has the form $(t, t^2, \ldots, t^d)$.
- If this point lies in $h$, we obtain $a_1t + a_2t^2 + \cdots + a_dt^d - b = 0$
- In this situation, $t$ is a root of a nonzero polynomial $p_h(t)$ of degree at most $d$, and thus the number of intersections of $h$ with $\gamma$ is at most $d$
- If there are $d$ distinct roots, then they must be all simple.
- At a simple root, the polynomial $p_h(t)$ changes sign, and this means that the curve $\gamma$ passes from one side of $h$ to the other

## Corollary

- Every $d$ points of the moment curve are affinely indepedent.
  - Otherwise, we could pass a hyperplane through them plus one more point of $\gamma$
- The moment curve readily supplies *explicit* examples of point sets in general position

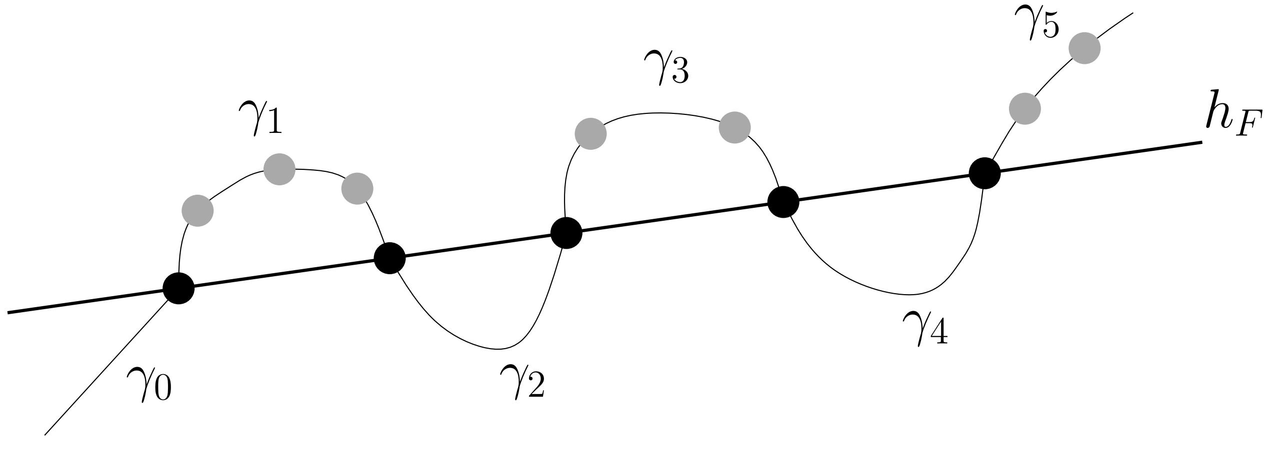
How many facets does a cyclic polytope have?

- Each facet is determined by a $d$-tuple of vertices
- Distinct $d$-tuples determine distinct facets.

Proposition (Gale's evenness criterion)
Let $V$ be the vertex set of a cyclic polytope $P$ considerd with the linear ordering $\leq$ along the moment curve (larger vertices have larger values of the parameter $t$).
Let $F = \{v_1, v_2, \ldots, v_d\} \subseteq V$ be a $d$-tuple of vertices of $P$, where $v_1 < v_2 < \cdots < v_d$.
Then $F$ determines a facet of $P$ if and only if for any two vertices $u, v \in V \setminus F$, the number of vertices $v_i \in F$ with $u < v_i < v$ is even.

Sktech of Proof.

- Let $h_F$ be the hyperplane affinely spanned by $F$
- $F$ determines a facet if and only if all the points of $V \setminus F$ lies on the same side of $h_F$
- Since the moment curve $\gamma$ intersects $h_F$ in exactly $d$ points, i.e., at the points of $F$, $\gamma$ is partitioned into $d+1$ pieses, say $\gamma_0, \ldots, \gamma_d$, each lying completely in one of the half-spaces.
- If the vertices of $V \setminus F$ are all contained in the odd-numbered pieces $\gamma_1, \gamma_3, \ldots$, or if they are all contained in the even-number pieces $\gamma_0, \gamma_2, \ldots$, then $F$ detemines a facet
- The above condition is equivalent to the Gale's criterion.



Theorem (Number of facets of a cyclic polytope)
The number of facets of a $d$-dimensional cyclic polytope with $n$ vertices is

$$\binom{n-\lfloor d/2\rfloor}{\lfloor d/2\rfloor}+\binom{n-\lfloor d/2\rfloor-1}{\lfloor d/2\rfloor-1} \text{ for } d \text{ even, and}$$

$$2\binom{n-\lfloor d/2\rfloor-1}{\lfloor d/2\rfloor} \text{ for } d \text{ odd.}$$

For fixed $d$, this has the order of magnitude $n^{\lfloor d/2\rfloor}$.

Sketch of Proof

- # facets = # ways placing $d$ black circles and $n-d$ white circles in a row where an even number of black circles lies between each two white circles.
- An arrangement of black and white circles are paired if any contiguous segment of black circles has an even length
- # paired arrangements of $2k$ black and $n-2k$ white circles is $\binom{n-k}{k}$
  - since by deleting every second black circles we get a one-to-one correspondence with selections of the position of $k$ black circles among $n-k$ possible position
- Consider an *odd* $d = 2k+1$. We must have an odd number of consecutive black circles at the begining or at the end (but not botth)
  - For the former case, delete the initial black circle, and get a paired arrangment of $2k$ black and $n-1-2k$ white circles
  - For the latter case, do symmetrically
  - This implies the first formula for odd $d$
- For $d = 2k$, # initial consecutive black circles is either odd or even.
  - In the even case, we have a paired arrangement, leading to $\binom{n-k}{k}$ possibilities
  - In the odd case,
    * We also have an odd number of consecutive black circles at the end
    * By deleting the first and last black circles we obtain a paired arrangement of $2(k-1)$ black circles and $n-2k$ white circles
    * This implies $\binom{n-k-2}{k-1}$ possibilities

**Upper Bound Theorem**
Among all $d$-dimensional convex polytopes, claims that the cyclic polytope has the largest possible number of faces.

**Asymptotic upper bound theorem** (Proposition 5.5.2)
A $d$-dimensional convex polytope with $n$ vertices has at most $2\binom{n}{\lfloor d/2 \rfloor}$ facets and no more than $2^{d+1}\binom{n}{\lfloor d/2 \rfloor}$ faces in total. For $d$ fixed, both quantities thus have the order of mangnitude $n^{\lfloor d/2 \rfloor}$.

**Proposition 5.5.3.**
Let $P$ be a $d$-dimensional simplicial polytope. Then

**(a)** $f_0(P) + f_1(P) + \cdots + f_d(P) \leq 2^d f_{d+1}(P)$, and

**(b)** $f_{d-1}(P) \leq 2f_{\lfloor d/2 \rfloor - 1}(P)$

This implies Proposition 5.5.2 for simplicial polytope

- The number of $(\lfloor d/2 \rfloor - 1)$-faces is certainly no bigger than $\binom{n}{\lfloor d/2 \rfloor}$, which is the number of all $\lfloor d/2 \rfloor$-tuples of vertices

Sketch of Proof

- Consider the dual polytope $P^*$, which is simple
- Need to prove $\sum_{k=0}^{d} f_k(P^*) \leq 2^d f_0(P^*)$ and $f_0(P^*) \leq 2f_{\lceil d/2 \rceil}(P^*)$.
- Each face of $P^*$ has at least one vertex, and every vertex of a simple $d$-polytope is incident to $2^d$ faces, which is gives the first inequality
- The remainng is to bound the number of vertices in terms of the number of $\lceil d/2 \rceil$-faces.
  - It shows where the mysterious exponent $\lfloor d/2 \rfloor$ comes from

Sketch of Proof (Continues)

- Rotate the polytope $P^*$ so that no two vertices share the $x_d$-coordinace
  - i.e., no two vertices have the same vertical level
- Consider a vertex $v$ with $d$ edges emanating from it.
- By the pigeonhole principle, there are at least $\lceil d/2 \rceil$ edges directed downwards or at least $\lceil d/2 \rceil$ edges directed upwards
- In the former case, every $\lceil d/2 \rceil$-tuple of edge going up determines a $\lceil d/2 \rceil$-face for which $v$ is the lowest vertex
- In the latter case, every $\lceil d/2 \rceil$-tuple of edge going down determines a $\lceil d/2 \rceil$-face for which $v$ is the highest vertex
- There exists at least one $\lceil d/2 \rceil$-face for which $v$ is the lowest vertex or the highest vertex
- Since the lowest vertex and the highest vertex are unqiue for each face, the number of vertices is no more than twice the number of $\lceil d/2 \rceil$-faces.

## Warning

- For simple polytopes, the total combinatorial complexity is proportional to the number of vertices
- For simplicial polytope, the combinatorial complexity is proportional to the number of facets
- For polytopes that are neigher simple nor simplicial, the number of faces of intermediate dimenions can have larger order of magnitude than both the number of facets and the number of vertices.

## Nonsimplicial polytopes

- To prove the asymptotic upper bound
- To use the perturbation argument

**Lemma**

For any $d$-dimensional convex polytope $P$, there exists a $d$-dimensional simplicail polytope $Q$ with $f_0(P) = f_0(Q)$ and $f_k(Q) \geq f_k(P)$ for all $k = 1, 2, \ldots, d$.

**Sketch of Proof**

- Basic Idea
  - Move (perturb) every vertex of $P$ by a very small amount, in such a way that the vertices are in general position
  - Show that each $k$-face of $P$ give rise to at least one $k$-face of the perturbed polytope.
- Process the vertices one by one
- Let $V$ be the vertex set of $P$ and let $v \in V$
- The operation of $\epsilon$-*pushing* $v$:
  - Choose a point $v'$ lying in the interior of $P$, at distance at most $\epsilon$ from $v$, and on no hyperplane determined by the points of $V$.
  - Set $V' = (V \setminus \{v\}) \cup \{v'\}$
- If we successively $\epsilon_v$-push each vertex of the polytope, the resulting vertex set is in general position and we have a simple polytope.
- Show for any polytope $P$ with vertex set $V$ and any vertex $v \in V$, there is an $\epsilon > 0$ such that $\epsilon$-pushing $v$ does not decrease the number of faces.
- Let $U \subset V$ be the vertex set of a $k$-face of $P$, $0 \leq k \leq d-1$, and let $V'$ arise from $V$ by $\epsilon$-pushing $v$.
- If $v \notin U$, then $U$ determines a face of conv$(V')$. so we assume $v \in V$.
- If $v$ lies in the affine hull of $U \setminus \{v\}$
  - Claim $U \setminus \{v\}$ determines a $k$-face of conv$(V')$.
  - This follows from Exercise 5.3.7: A subset $U \subset V$ is the vertex of a face of conv$(V)$ if and only if the affine hull of $U$ is disjoint from conv$(V \setminus U)$.

Sketch of Proof (continues)

- If $v$ lies outside the affine hull of $U \setminus \{v\}$:
  - Show $U' = (U \setminus \{v\}) \cup \{v'\}$ determines a $k$-face of $\mathrm{conv}(V')$
  - The affine hull of $U$ is disjoint from the compact set $\mathrm{conv}(V \setminus U)$
  - If we move $v$ continuously by a sufficiently smal amount, the affine hull of $U$ moves continuously, so there is an $\epsilon > 0$ such that if we move $v$ within $\epsilon$ from its original position, the considered affine hull and $\mathrm{conv}(V \setminus U)$ remain disjoint.