An Optimal Competitive Strategy for Looking Around a Corner

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Abstract

We consider a problem of motion planning under uncertainty. A robot can navigate freely in the plane and, using a built-in vision system, can determine distances and angles. Initially, the robot stands at a point close to a an edge of a polygonal obstacle (e. g. a wall of a huge building) and faces a corner at distance 1 from its position. The other wall which forms the corner is invisible from the starting position and the robot does not know the angle of the corner. The task of the robot is to move on a short path to a point where that wall becomes visible.

We show that there is a competitive strategy which guarantees that, for any possible value of the angle, the length of the path the robot walks until it can look around the corner is bounded by the length of the shortest path to do so, times the constant $c \approx 1.21218$. Furthermore, we prove that our strategy is optimal in that no smaller competitive factor than c can be achieved. We give a simple formula for the robot to find the optimal path.

A more general problem arises if the robot's starting point is not required to lie directly at the visible wall. We provide optimal competitive strategies for all such cases; the competitive factor varies between 1 and c, depending on the angle between the visible wall and the line through the starting point and the corner.

Key words. Motion planning, navigation, competitive algorithms, uncertainty, robotics.

1 Introduction

Algorithmic motion-planning in robotics is a classical field in computational geometry, see Schwartz and Sharir [11], Schwartz and Yap [12], or Mitchell [8] for surveys.

In the majority of the existing work it is assumed that the environment in which the system moves is known in advance. In real life, this assumption is not always granted. Autonomous vehicles should be able to find their ways through, or learn,

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unknown terrain as efficently as possible. This means that the task must be accomplished correctly, but as little as possible of resources like time or energy should be used.

In the last years, several researchers independently began to apply to geometric planning problems the concept of *competitive algorithms* introduced by Sleator and Tarjan [13]. Here one compares what can be achieved with incomplete information against what could be achieved if full information was available. More precisely, S is a *competitive strategy* for problem class $\mathcal{P}$ if there exists a constant c such that, for each instance P of $\mathcal{P}$, the cost of applying S to P does not exceed c times the cost of solving P in an optimal way, given full information. The minimum c satisfying this condition is called the *competitive factor* of S.

Among other work, competitive geometric algorithms have been developed by Papadimitriou and Yanakakis [10], Blum, Raghhavan, and Schieber [2], and Eades, Lin, and Wormald [5] for path planning in the presence of obstacles in the plane, by Deng, Kameda, and Papadimitriou [4] for learning the interior of a polygon that may have a bounded number of holes, and by Klein [7] for finding a path in the interior of special simple polygons called streets.

Figure 1: The shortest path from W to $M(\varphi)$.

In this paper we study an elementary problem related to learning an unknown environment. Suppose that two halflines meet at the origin, O, as shown in Figure 1. The shaded wedge formed by the halflines is opaque and has an angle less than 180°. Now let us assume that on one of the halflines a mobile robot is located at point W, outside the wedge, that is equipped with an on-board vision system facing O. Its task is to "look around the corner", i. e. to inspect the other halfline which is invisible from W (but for its endpoint, O). This task might occur as one basic step for an autonomous system which explores an unknown environment. A strategy for this class of problems should produce a path from W to any point on the prolongation, $M(\varphi)$, of the invisible halfline. Here, φ denotes the angle between the invisible halfline and the prolongation of the visible one.

A *competitive* strategy for this problem should have a bounded relative detour, which is the ratio of the robot's path length and the shortest path. If the value of φ lies between 0° and 90°, the shortest path from W to $M(\varphi)$ is perpendicular to

$M(\varphi)$, see Figure 1 (i). If φ is bigger than $90°$, the point of $M(\varphi)$ closest to W is the corner itself; see (ii).

The only uncertainty for the robot in this type of problem is the actual value of φ. Obviously, walking straight to the corner always fulfills the task. But the length of the path created is a constant, whereas an arbitrarily short path could be sufficient for small values of φ. This shows that walking in a fixed direction does *not* lead to a competitive strategy.

In Section 2 we characterize, by means of a quite natural property, the competitive strategies among all strategies for the corner problem. Although it turns out that there is no lack of competitive strategies, it is not so easy to construct a strategy whose factor is—say—less than 1.5. A lower bound for the competitive factor of all strategies for this problem is given in Section 3.

In Section 4 we start attacking the optimality problem. Our approach leads to the differential equation

$$w' = (w^2 + 1)(1 - w \cot x)$$

that is subject to certain additional constraints. Although this equation is of Abelian type, a closed-form solution is apparently not provided by the theory. But we can show that the required solution must exist, which is not quite straightforward, due to the additional requirements to be met.

From the bare existence, and from the functional properties of the differential equation, we are able to derive that the solution of the differential equation above leads to a competitive strategy whose factor equals $1.21218\ldots$, and that no better strategy exists. This will be shown in Section 5. The key step is in proving that the curve implied by this strategy is convex. While the analysis of our strategy and the proof for optimality are rather complicated and use means from the theory of ordinary differential equations, the resulting strategy is surprisingly simple.

The problem can be generalized to the situation in which the robot's starting point, W, does not lie on a wall but in the free area outside the wedge. The problem changes, because now the unknown angle φ can take on its values only in a smaller range. In Section 6 we show how to construct an optimal competitive strategy for each possible value of the angle, β, between the visible wall and the line through the starting point and the corner. It turns out that the competitive factor is strictly decreasing from $c = 1.21218\ldots$ to 1.0, as β is increased. For $\beta = \frac{\pi}{2} = 90°$, for example, the optimal factor is approximately 1.1261.

In Section 7, we will see a simple rule for the robot to compute the walking direction for the optimal path. Furthermore, an easy-to-compute approximation is given, whose competitive factor is only 3.1 % worse than the optimum.

2 Simple Competitive Strategies

As shown in Figure 1, we place the corner at the origin of the coordinate system. The robot's starting point, W, lies on the negative y-axis at distance 1 from O.

Let φ be the angle between the invisible wall and the prolongation, $M(\pi)$, of the visible wall. The angle φ is between 0 and π. By $a(\varphi)$ we denote the distance between W and $M(\varphi)$. As mentioned in the introduction, we have

$$a(\varphi) = \begin{cases} \sin\varphi & : \quad 0 \le \varphi \le \frac{\pi}{2} \\ 1 & : \quad \frac{\pi}{2} < \varphi \le \pi \end{cases}$$

Note that a is continuously differentiable.

A *strategy* for our problem is a curve that starts at point W on the visible wall and leads to the prolongation of the visible wall. In fact, for each possible value of φ there is a point on such a curve from where the other wall is visible, namely the intersection with $M(\varphi)$.

Let $A_S(\varphi)$ be the length of the path generated by strategy S between W and the first point of intersection with $M(\varphi)$. The *competitive function*, $f_S(\varphi)$, of S is the ratio of $A_S(\varphi)$ and $a(\varphi)$, and its *competitive factor*, c_S, is the maximum value of $f_S(\varphi)$.

$$f_S(\varphi) = \frac{A_S(\varphi)}{a(\varphi)}\,, \qquad c_S = \sup_{\varphi \in (0,\pi]} f_S(\varphi)$$

By $f_S(0)$ we mean $\lim_{\varphi\to 0} f_S(\varphi)$, if it exists. The problem is to find a strategy whose competive factor is as small as possible.

If a strategy reaches a halfline $M(\varphi')$ for the first time, turns back and eventually reaches $M(\varphi')$ again then this part of the path can be cut off and replaced by a radial line segment, improving on the strategy. Strategies including radial line segments can in turn be approximated arbitrarily closely by strategies that can be described by a system of polar coordinates about O.

Definition 1 A curve $S = \big(\varphi, s(\varphi)\big)$ in polar coordinates about O is called a *strategy* for the corner problem if the following holds.

(i) s is a continuous function on an interval $[0, \sigma]$, where $\sigma \le \pi$.

(ii) On the open interval $(0, \sigma)$, s is piecewise continuously differentiable and $s'(0)$ exists (possibly $\pm\infty$).

(iii) $s(0) = 1$.

(iv) If $s(\sigma) \ne 0$, then $\sigma = \pi$.

The last property states that S must arrive at $M(\pi)$, including the corner.

First, we show that each sensible strategy is in fact competitive.

Lemma 2 *Let $S = \big(\varphi, s(\varphi)\big)$ be a strategy. Then S is competitive iff $|s'(0)| < \infty$. The estimation*

$$c_S \ge \sqrt{s'^2(0) + 1}$$

holds for the competitive factor.

Proof. Since

$$A_S(\varphi) = \int_0^{\varphi} \sqrt{s'^2(t) + s^2(t)}\, dt \tag{1}$$

holds for the arc length of a curve in polar coordinates, we obtain from de l'Hospital's theorem (i. e. by taking derivatives in both numerator and denominator)

$$c_S \geq f_S(0) = \lim_{\varphi \to 0} \frac{A_S(\varphi)}{\sin\varphi} = \lim_{\varphi \to 0} \frac{\sqrt{s(\varphi)^2 + s'^2(\varphi)}}{\cos\varphi} = \sqrt{s'^2(0) + 1}$$

Since f_S is a continuous function on the interval $[0, \sigma]$, it takes on its maximum value. □

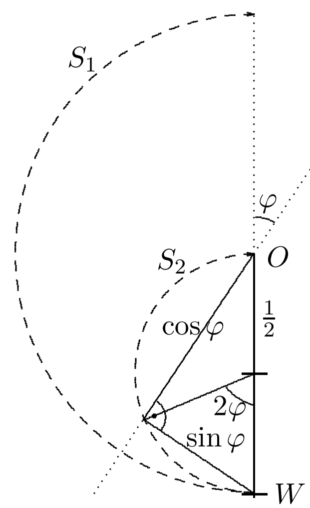

Figure 2: Simple strategies achieving competitive factors $c_{S_1} = \pi$ and $c_{S_2} = \frac{\pi}{2}$.

To give an example, consider strategy S_1 that walks along the circle through W with center in the corner, i. e. $s_1(\varphi) = 1$ for all φ, see Figure 2. We have $A_{S_1}(\varphi) = \varphi$ and $f_{S_1}(\varphi) = \varphi/\sin\varphi$ for $\varphi \in [0, \frac{\pi}{2}]$, $f_{S_1}(\varphi) = \varphi$ for $\varphi \in [\frac{\pi}{2}, \pi]$.

It is easy to check that f_{S_1} attains its maximum at $\varphi = \pi$, thus $c_{S_1} = \pi \approx 3.14159$.

A better strategy, S_2, is the following. We walk along the circle with radius $\frac{1}{2}$ through W centered at the mid point between W and O, i. e. $s_2(\varphi) = \cos\varphi$. Then $s_2(\frac{\pi}{2}) = 0$, i. e. we reach the corner, so we only need to consider the angles φ in the interval $[0, \frac{\pi}{2}]$.

But $A_{S_2}(\varphi) = \frac{1}{2}(2\varphi)$ holds, implying $f_{S_2}(\varphi) = f_{S_1}(\varphi)$ for $\varphi \in [0, \frac{\pi}{2}]$. The maximum value is only $\frac{\pi}{2} \approx 1.5708$.

Remark. It is interesting to observe that two quite different strategies, S_1 and S_2, have nearly identical functions f_{S_1} and f_{S_2}.

3 A Lower Bound

A simple lower bound is obtained in the following way, see Figure 3.

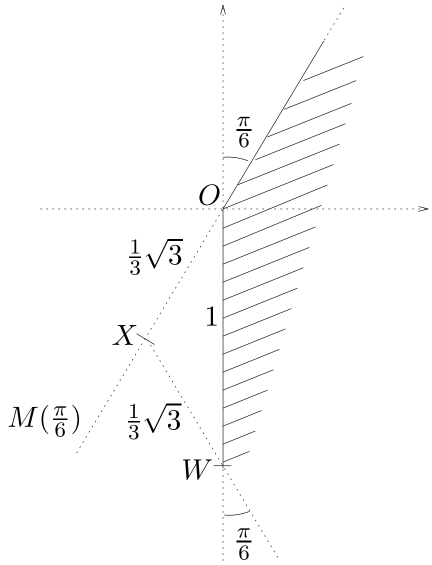

Figure 3: Deriving a lower bound.

We fix an angle of $\frac{\pi}{6} = 30°$, and on the ray $M(\frac{\pi}{6})$ we mark the point X that has distance $\frac{1}{3}\sqrt{3}$ from the corner, O, as well as from W. Now we consider an arbitrary strategy S.

It must, at some stage, arrive at $M(\frac{\pi}{6})$. If it hits the line to the left of X then we define angle φ to be almost π. So, the robot still has to walk at least to the corner, and the length, $A_S(\pi)$, is at least as big as the distance from W to X plus the distance from X to the corner, the shortest path being of length 1, which results in $c_S \geq \frac{2}{3}\sqrt{3}$.

If the path hits the dotted line to the right of X then we let $\varphi = \frac{\pi}{6}$, and the path length, $A_S(\frac{\pi}{6})$, is at least the distance from W to X, whereas the shortest path measures $\sin \frac{\pi}{6} = \frac{1}{2}$, which again comes to

$$c_S \geq f_S(\frac{\pi}{6}) = \frac{A_S(\frac{\pi}{6})}{\frac{1}{2}} \geq \frac{2}{3}\sqrt{3} .$$

So, we have a gap between the lower bound of $\frac{2}{3}\sqrt{3} \approx 1.1547$ and the upper bound of $\frac{\pi}{2} \approx 1.5708$ of Section 2. In principle, it is not clear why it should always be possible to close a gap like this. However, for this special problem there is in fact a strategy whose competitive factor is provably optimal.

4 A Differential Equation

Intuitively, if one tries to improve on a given strategy S by modifying it such that the maximum value for $f_S(\varphi)$ becomes smaller, some other values $f_S(\varphi')$ will increase. The key idea towards an optimal strategy is to assume that this process can reach a state of equilibrium, and to look for a strategy R such that $f_R(\varphi) = c$, i. e. constant, *for all* φ. This constant, c, would then be the competitive factor of the strategy. If there is more than one strategy with this property, we would look for the one with the smallest value of c.

Since $a(\varphi) = \sin\varphi$ for $\varphi \in [0, \frac{\pi}{2}]$, we try to solve the following equation.

$$f_R(\varphi) = \frac{A_R(\varphi)}{\sin\varphi} = c \quad \text{for all } \varphi \in [0, \frac{\pi}{2}] \tag{2}$$

After inserting (1) of Lemma 2 into (2), multiplying the result by $\sin\varphi$, and taking the derivative with respect to φ, we obtain

$$c\cos\varphi = A'_R(\varphi) = \sqrt{r'^2(\varphi) + r^2(\varphi)} \tag{3}$$

This is an ordinary differential equation for the unknown function r, the initial condition is $r(0) = 1$ because we have to start from W with angle $\varphi = 0$. Since we want the robot to eventually arrive at the corner, the solution should exist on an interval $[0, \sigma]$ where $r(\sigma) = 0$ holds. For $\varphi \in [0, \sigma)$ the radius $r(\varphi)$ should be strictly positive.

Some transformations of this equation are appropriate. First, we solve Equation 3 for $r'(\varphi)$.

$$r'(\varphi) = -\sqrt{c^2\cos^2\varphi - r^2(\varphi)} \tag{4}$$

The negative square root is taken because the solution y should be decreasing, meaning that the robot should always come closer to the corner, as it moves along R. An increasing function would be even worse than the circular strategy from Section 2.

Now we can eliminate the constant c from the equation by formally replacing $r(\varphi)$ by $cu(\varphi)$. The initial condition changes to $u(0) = \frac{1}{c}$.

$$u'(\varphi) = -\sqrt{\cos^2\varphi - u^2(\varphi)} \tag{5}$$

Since c is the competitive factor of the solution $\big(\varphi, r(\varphi)\big)$, our problem now looks as follows.

Problem. Find the minimum $c > 1$, such that the ordinary differential equation (5) has a solution on some interval $[0, \sigma] \subseteq [0, \frac{\pi}{2}]$ subject to the following constraints:

$$u(0) = \frac{1}{c}$$

$$u(\varphi) > 0 \quad \text{for } \varphi \in [0, \sigma)$$

$$u(\sigma) = 0$$

Any real valued solution must satisfy $|u(\varphi)| \leq \cos\varphi$ because of the square root in (5). As we will see later, for c to be minimal, σ must be as large as possible, i. e. $\sigma = \frac{\pi}{2}$.

In the following, it will be shown that this heuristic approach works. In this section, the existence of a solution of the above problem in which $\sigma = \frac{\pi}{2}$, $c \approx 1.21218$ is proven. Section 5 shows that this solution, R, is in fact optimal among *all* possible strategies.

Remark. Equation 5 can be transformed into an equation of Abelian type.

$$w'(x) = \Big(w^2(x) + 1\Big)\Big(1 - w(x)\cot x\Big) \tag{6}$$

However, there is no complete theory on equations of this type, and a closed-form solution seems not to be known, see Kamke [6] or Murphy [9].

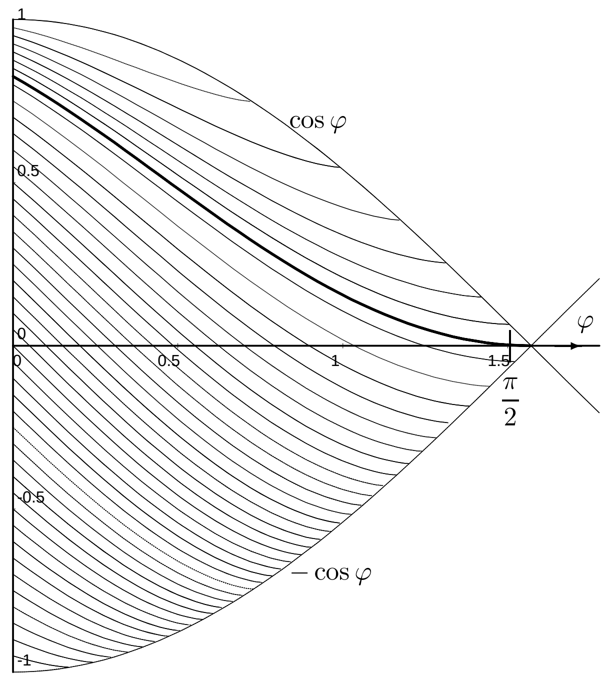

Figure 4: Solutions of the differential equation for several initial values.

Figure 4 shows the numerical solutions $u(\varphi)$ of Equation 5 in the domain $0 \leq \varphi \leq \frac{\pi}{2}$, $|u(\varphi)| \leq \cos\varphi$. The polar coordinates $\big(\varphi, u(\varphi)\big)$ of the solutions are represented in a Cartesian coordinate system. Only the positive parts of the solutions have a meaning in our original problem. The figure already suggests the existence of a solution of (5) satifying $u(\frac{\pi}{2}) = 0$, and a precise proof of this follows.

Let $f(t, u) = -\sqrt{\cos^2 t - u^2}$ be the right hand side of (5). Let D be the open region $\{(t, u) | 0 < t < \frac{\pi}{2}, |u| < \cos t\}$; f is defined on the closure of D. First, we study (5) without the additional constraints stated in the problem. We parametrize those solutions that intersect the abscissae in Figure 4 by their points of intersection.

Lemma 3 *For each $\varphi \in (0, \frac{\pi}{2})$ there is a unique solution u_φ of Equation 5 with $u_\varphi(\varphi) = 0$. The solution u_φ exists in an interval $[0, k_\varphi]$ where $\varphi < k_\varphi < \frac{\pi}{2}$ and $u_\varphi(k_\varphi) = -\cos k_\varphi$.*

Proof. The function f is continuously differentiable with respect to u and therefore fulfills a local Lipschitz condition on y, i. e. for $(t_0, u_0) \in D$ there is a neighbourhood $N \subset D$ of (t_0, u_0) and a constant L such that

$$|f(t,u) - f(t,\overline{u}) \leq L|u - \overline{u}| \quad \text{for all } (t,u), (t,\overline{u}) \in N$$

From a main theorem of the theory of ordinary differential equations it follows that, for any $(t, v) \in D$, Equation 5 with the initial condition $u(t) = v$ has a unique solution that extends to the boundary of D. In particular, we have a unique solution u_φ for $u(\varphi) = 0$.

Let k_φ be the maximum value for which u_φ is defined. Since the function f is strictly negative in D, the solution u_φ is strictly decreasing. Thus $u_\varphi(k_\varphi) < 0$, and because $\big(k_\varphi, u_\varphi(k_\varphi)\big)$ must be on the boundary of D, we have $u_\varphi(k_\varphi) = -\cos k_\varphi$.

Assume that u_φ, on the left side of the vertical line through φ, does not extend to $t = 0$. Then u_φ must hit the upper cosine wave. So there exists an $l \in (0, \varphi)$ with $u_\varphi(l) = \cos l$ and $u_\varphi(t) < \cos t$ for $t > l$. Then

$$\begin{aligned} 0 &= \sqrt{\cos^2 l - u_\varphi^2(l)} \\ &= u'_\varphi(l) = \lim_{\delta \to 0} \frac{u_\varphi(l+\delta) - u_\varphi(l)}{\delta} \\ &\leq \lim_{\delta \to 0} \frac{\cos(l+\delta) - \cos l}{\delta} = -\sin l \\ &< 0 \end{aligned}$$

which is a contradiction. □

Since $u'_\varphi(k_\varphi) = 0$, we can extend u_φ to the whole interval $[0, \frac{\pi}{2}]$ by defining

$$u_\varphi(t) = -\cos k_\varphi \qquad \text{for } t > k_\varphi$$

such that the extended function is continuously differentiable in $[0, \frac{\pi}{2}]$. From now on, we use u_φ for the extended function.

Lemma 4 *There is a unique solution for Equation 5 with $u(\frac{\pi}{2}) = 0$ which also satisfies the constraints of the problem.*

Proof. Let φ_n be a strictly increasing sequence converging to $\frac{\pi}{2}$, for example $\varphi_n = \frac{\pi}{2} - \frac{1}{n}$, and, for brevity, let $u_n = u_{\varphi_n}$.

For some fixed $t \in [0, \frac{\pi}{2}]$, we consider the sequences $u_n(t)$ and $u'_n(t)$. It follows from the uniqueness of the solutions of Equation 5 that u_n and u_{n+1} do not cross, so $u_n(t)$ is a strictly increasing sequence which is bounded from above by $\cos t$. It is then clear that $u(t) = \lim_{n\to\infty} u_n(t)$ exists. In particular, we have $u(\frac{\pi}{2}) = 0$.

If $t < \frac{\pi}{2}$ we choose an n_0 such that $\varphi_{n_0} > t$. Then we have $u_n(t) > 0$ for all $n \geq n_0$ and

$$\begin{aligned} u'_{n+1}(t) - u'_n(t) &= \sqrt{\cos^2 t - u_n^2(t)} - \sqrt{\cos^2 t - u_{n+1}^2(t)} \\ &= \frac{u_{n+1}^2(t) - u_n^2(t)}{\left(\sqrt{\cos^2 t - u_n^2(t)} + \sqrt{\cos^2 t - u_{n+1}^2(t)}\right)} \\ &> 0 \end{aligned}$$

It follows that $u'_n(t)$ is a strictly increasing sequence for $n \geq n_0$ (but note that n_0 depends on t). For $t = \frac{\pi}{2}$ we have the constant sequence $u'_n(\frac{\pi}{2}) = 0$. In either case $u'_n(t)$ is bounded since $-1 \leq f(t, u) \leq 0$. So the function $g(t) = \lim_{n\to\infty} u'_n(t)$ is well-defined, independently of the choice of φ_n. In particular, we have $g(\frac{\pi}{2}) = 0$.

The sequence of functions u_n is equicontinuous because of

$$|u_n(t) - u_n(\overline{t})| = |\int_{\overline{t}}^{t} u'_n(s)\, ds| \leq |t - \overline{t}|$$

According to the theorem of Ascoli-Arzelà [14] and due to the monotonicity of u_n it follows that u_n is *uniformly* convergent to u, hence u must be continuous.

Then the function g is continuous, too, because of

$$g(t) = \lim_{n\to\infty} u'_n(t) = -\lim_{n\to\infty} \sqrt{\cos^2 t - u_n^2(t)} = -\sqrt{\cos^2 t - u^2(t)}$$

For an arbitrarily small $\varepsilon > 0$, we consider the interval $[0, \frac{\pi}{2} - \varepsilon]$. There exists an n_0 such that for $n \geq n_0$ all sequences $u'_n(t)$ are strictly increasing. Since the limit function, g, is continuous, a theorem of Dini's [1] applies and shows that u'_n converges uniformly to g. It follows from a well-known result from calculus that, on $[0, \frac{\pi}{2} - \varepsilon]$, u is differentiable and $u' = g$ holds.

Since $u'(t) = g(t) = -\sqrt{\cos^2 t - u^2(t)}$, the function u is a solution of Equation 5 on $[0, \frac{\pi}{2})$, but since any such solution extends to the boundary of D and $u(\frac{\pi}{2}) = 0$, the assertion follows. □

Let u be the solution provided by Lemma 4. If we put $r(\varphi) = c \cdot u(\varphi)$, where $c = 1/u(0)$, we obtain a solution $R = \left(\varphi, r(\varphi)\right)$ of our original Equation 4. By construction, $r(0) = 1$, $r(\frac{\pi}{2}) = 0$, and for each $\varphi \in [0, \frac{\pi}{2}]$, the ratio $f_R(\varphi)$ is the constant c. The value of c has been determined to be approximately 1.21218, using numerical methods.

The transformed curves depicted in Figure 5 correspond to the nonnegative solutions shown in Figure 4. It is easy to see that R is optimal within the class of strategies S that have constant ratio $f_S(\varphi)$.

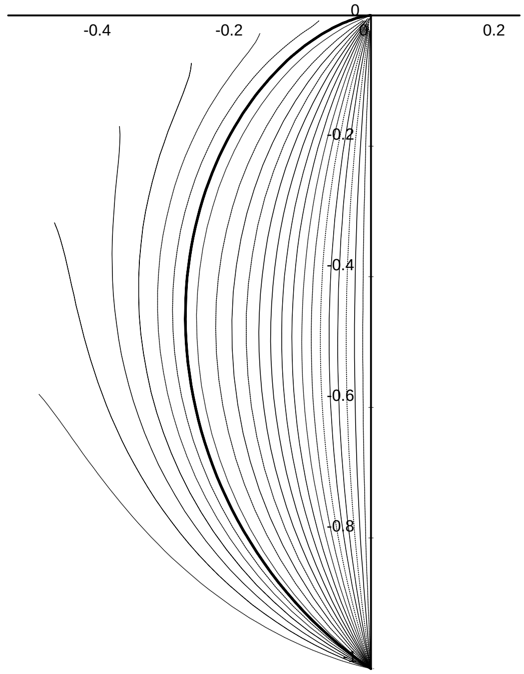

Figure 5: Solutions of Equation 4 corresponding to the curves in Figure 4, the thick path represents the optimal solution.

5 The Optimal Strategy

Now we prove that strategy R, as described in the preceding section, is in fact optimal among all competitive strategies for the corner problem.

Lemma 5 *The strategy $R = \Big(\varphi, r(\varphi)\Big)$ forms a convex curve.*

Proof. We use the curvature formula of a curve in polar coordinates.

$$\kappa = \frac{r^2 + 2r'^2 - rr''}{(r'^2 + r^2)^{\frac{3}{2}}}$$

The curve is convex iff the curvature κ is strictly positive. The sign is determined by the nominator. We use that $r = c \cdot u$ holds, where u is the solution of the differential equation (5), from which we get $u' = -\sqrt{\cos^2\varphi - u^2}$ and $u'' = -u - \sin\varphi\cos\varphi/u'$. Hence,

$$\begin{aligned}
r^2 + 2r'^2 - rr'' &= c^2\left(u^2 + 2(\cos^2\varphi - u^2) + u^2 + \frac{u}{u'}\sin\varphi\cos\varphi\right)\\
&= c^2\cos\varphi\left(2\cos\varphi + \frac{u}{u'}\sin\varphi\right)\\
&= c^2\cos\varphi\frac{2\cos\varphi\sqrt{\cos^2\varphi - u^2} - u\sin\varphi}{\sqrt{\cos^2\varphi - u^2}}\\
&= \frac{c^2\cos\varphi\left(4\cos^2\varphi(\cos^2\varphi - u^2) - u^2\sin^2\varphi\right)}{\sqrt{\cos^2\varphi - u^2}\left(2\cos\varphi\sqrt{\cos^2\varphi - u^2} + u\sin\varphi\right)}\\
&= \frac{c^2\cos\varphi\,(4\cos^4\varphi - u^2(1 + 3\cos^2\varphi))}{\sqrt{\cos^2\varphi - u^2}\left(2\cos\varphi\sqrt{\cos^2\varphi - u^2} + u\sin\varphi\right)}
\end{aligned}$$

For $\varphi \in [0, \frac{\pi}{2}]$ we have that $\kappa > 0$ if and only if

$$\begin{aligned} 0 &< 4\cos^4\varphi - u^2(1+3\cos^2\varphi) \\ u^2 &< \frac{4\cos^4\varphi}{1+3\cos^2\varphi} \\ u &< \frac{2\cos^2\varphi}{\sqrt{1+3\cos^2\varphi}} \end{aligned}$$

We call $v(\varphi) = 2\cos^2\varphi/\sqrt{1+3\cos^2\varphi}$ the right hand side of the last inequality. It is clear that $0 < v(\varphi) < \cos\varphi$ for $\varphi \in (0, \frac{\pi}{2})$ and $v(0) = 1 > u(0)$, $v(\frac{\pi}{2}) = 0 = u(\frac{\pi}{2})$. We want to show that $u(\varphi) < v(\varphi)$ for $\varphi \in (0, \frac{\pi}{2})$.

Assume that $u(\omega) = v(\omega)$ for some $\omega \in (0, \frac{\pi}{2})$. Then

$$\begin{aligned} u'(\omega) - v'(\omega) &= -\sqrt{\cos^2\omega - v^2(\omega)} + \frac{(4+6\cos^2\omega)\sin\omega\cos\omega}{(1+3\cos^2\omega)^{\frac{3}{2}}} \\ &= \frac{(3+3\cos^2\omega)\sin\omega\cos\omega}{(1+3\cos^2\omega)^{\frac{3}{2}}} \\ &> 0 \end{aligned}$$

From $u'(\omega) > v'(\omega)$, we conclude that u must cross v from below at any point $\omega \in (0, \frac{\pi}{2})$ where they intersect, hence at most one such point can exist. Let $\psi \in (\omega, \frac{\pi}{2})$, and let q be a solution of Equation 5 such that $q(\psi) = v(\psi)$. To ψ the same argument as to ω applies, i. e. q crosses v from below, and this is the only intersection in $(\omega, \frac{\pi}{2})$. Since q has to stay below u and above v in $(\psi, \frac{\pi}{2})$ it follows $q(\frac{\pi}{2}) = 0$. But $q(\frac{\pi}{2}) = 0 = u(\frac{\pi}{2})$ contradicts the uniqueness of u stated in Lemma 4. $\square$

Theorem 6 *The strategy R as described at the end of Section 4 is an optimal competitive strategy for the corner problem.*

Proof. Let $S = \big(\varphi, s(\varphi)\big)$ be a strategy different from R.
We apply Lemma 2. If $|s'(0)| = \infty$ then S is not competitive.
Case 1. If $s'(0) \leq r'(0) = -\sqrt{c^2-1}$ then

$$c_S \geq \sqrt{s'^2(0)+1} \geq \sqrt{r'^2(0)+1} = c$$

In the remaining cases $s'(0) > r'(0)$ holds. Then there exists an angle ψ such that $s(\varphi) > r(\varphi)$ for $\varphi \in (0, \psi]$.
Case 2. There exists an angle $\chi \leq \frac{\pi}{2}$ such that $s(\chi) = r(\chi)$ and $s(\varphi) > r(\varphi)$ on $(0, \chi)$, see Figure 6. Then $A_S(\chi) > A_R(\chi)$, due to the convexity of R shown in Lemma 5. Hence,

$$c_S \geq f_S(\chi) = \frac{A_S(\chi)}{\sin\chi} > \frac{A_R(\chi)}{\sin\chi} = c$$

Case 3. $s(\varphi) > r(\varphi)$ for $\varphi \in (0, \frac{\pi}{2}]$. Then c_S is not less than the total arc length of S, which is bigger than the length of S between $\varphi = 0$ and $\varphi = \frac{\pi}{2}$, plus the length of the line segment from $\big(\frac{\pi}{2}, s(\frac{\pi}{2})\big)$ to the origin. The length of this curve, in turn, is bigger than $A_R(\frac{\pi}{2}) = c$, again by convexity of R; see Figure 6. $\square$

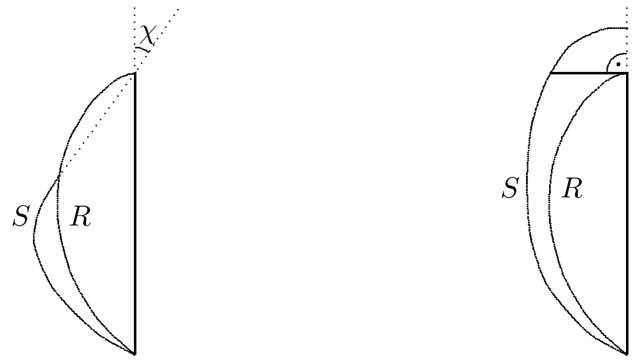

Figure 6: Case 2 and Case 3 of Theorem 6.

6 The General Corner Problem

A more general corner problem results if the robot's starting point, W, does not lie on a wall but in the free area outside the wedge. As before, let the corner of the wedge be at the origin, O, and let the distance between W and O be 1. Let β denote the angle between the visible wall and the line from W to O, see Figure 7; the problem from the previous sections is the one with $\beta = 0$. The problem becomes different, because now the unknown angle φ can take on its values only in $(0, \pi - \beta)$. This restriction is known to the robot who "sees" then angle β. In this section it is shown how to construct an optimal competitive strategy for each possible value of β.

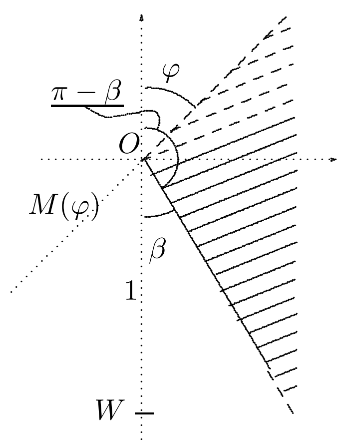

Figure 7: The general corner problem.

A *strategy* for the general corner problem with angle β is the same as a strategy for the corner problem, see Definition 1, except that the curve needs only arrive at $M(\pi - \beta)$, and not at $M(\pi)$.

To obtain a good strategy for values of β greater than $\frac{\pi}{2}$, one could think of the following. Varying the constant c in the initial condition $u(0) = 1/c$ of the differential equation (4), there is exactly one such c such that the solution of (4) stops at $\varphi = \pi - \beta$;

this solution corresponds to one of the leftmost curves of Figure 5. The larger β is the smaller the competitive factor c will be, and we have $c \downarrow 1$ for $\beta \uparrow \pi$. For $\beta = \frac{3}{4}\pi$, for example, the corresponding curve is competitive with factor $c \approx 1.03059$. But there can not be a proof for optimality, analogous to Theorem 6, since these curves are not convex at the end. And in fact, these strategies turn out to be *not* optimal. However, there is the following observation.

Observation. *For each $\beta \in (0, \pi)$, there is a unique strategy, $R_\beta = \big(\varphi, r_\beta(\varphi)\big)$, for the general corner problem that has the following properties. (The competitive factor of R_β is called c_β, the competitive function is f_β.)*

- *The strategy is composed of two parts, one from 0 to some angle $\delta_\beta \in (0, \pi - \beta]$, and one from δ_β to $\pi - \beta$.*
- *The first part of the curve has a constant competitive factor, i.e. $f_\beta(\varphi) = c_\beta$ for $\varphi \in [0, \delta_\beta]$. It is part of a solution of the differential equation (4) for $c = c_\beta$.*
- *The second part from δ_β to $\pi - \beta$ is a straight line segment perpendicular to $M(\pi - \beta)$.*
- *The function f_β takes on its maximum value also at the end, i.e. $f_\beta(\pi - \beta) = c_\beta$.*
- *The curve of R_β is continuously differentiable and convex.*
- *The competitive factor c_β is decreasing to 1 as β increases to π.*

A formal proof is not given here, but inspections with highly accurate numerical solutions of the differential equation strongly indicate that such curves always exist. See Figure 8 for an example with $\beta = \frac{\pi}{4}$.

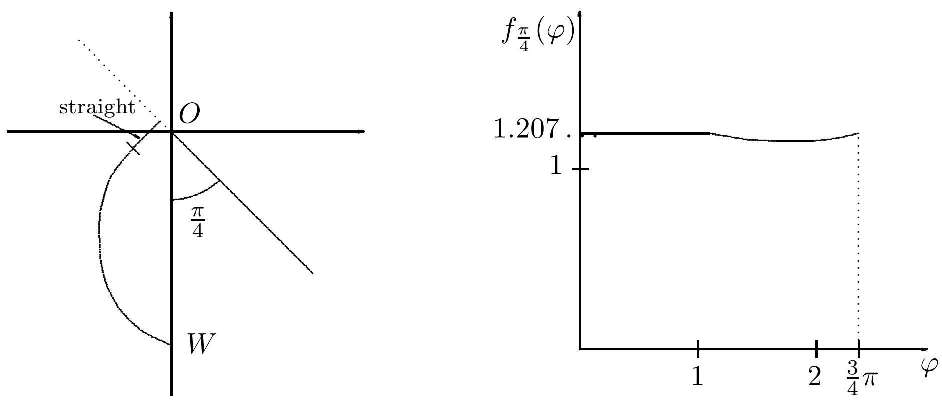

Figure 8: The optimal strategy and the corresponding competitive function for $\beta = \frac{\pi}{4} = 45°$.

Theorem 7 *For a fixed $\beta \in (0, \pi)$, the strategy R_β as described above is an optimal competitive strategy for the general corner problem with angle β, assuming the observation above holds.*

Proof. The convexity of our strategy R_β is essential, as well as the fact that it arrives perpendicularly at $M(\pi - \beta)$ and that $f_\beta(\varphi) = c_\beta$ holds for $\varphi \in [0, \delta_\beta] \cup \{\pi - \beta\}$.

The arguments are very similar to the proof of Theorem 6. Let $S = \big(\varphi, s(\varphi)\big)$ be a strategy different from R_β.

We apply Lemma 2. If $|s'(0)| = \infty$ then S is not competitive.

Case 1. $s'(0) \leq r'_\beta(0) = -\sqrt{c_\beta^2 - 1}$. This is identical to Case 1 of Theorem 6.

Case 2. There exists an angle $\chi \leq \delta_\beta$, i.e. on the first part of R_β, such that $s(\varphi) > r_\beta(\varphi)$ on $(0, \chi)$ and $s(\chi) = r_\beta(\chi)$. The arguments of Case 2 of Theorem 6 apply.

Case 3. There exists an angle $\chi > \delta_\beta$, i.e. on the second part of R_β, such that $s(\varphi) > r_\beta(\varphi)$ on $(0, \chi)$ and $s(\chi) = r_\beta(\chi)$. Then the total arc length of S, $A_S(\pi - \beta)$, is greater than the total arc length of R_β, $A_{R_\beta}(\pi - \beta)$, since $A_S(\chi) > A_{R_\beta}(\chi)$ due to the convexity of R_β, and the part of R_β after χ is the straight line perpendicular to $M(\pi - \beta)$, i.e. the shortest path from the point $\big(\chi, r_\beta(\chi)\big)$ to the line $M(\pi - \beta)$, so the part of S after χ can not be shorter.

Case 4. $s(\varphi) > r_\beta(\varphi)$ for $\varphi \in (0, \pi - \beta]$. Then the total arc length of S is greater than the total arc length of R_β, due to the convexity of R_β and to the fact that R_β arrives perpendicularly at $M(\pi - \beta)$. To see this, concatenate both curves with their reflections at $M(\pi - \beta)$. □

7 Conclusions

In the preceding sections, we have seen how the optimal strategies for the corner problem look like. For practical applications one might prefer approximations given in a closed form. For example, a good practical approximation Q for the optimal solution for $\beta = 0$ is given by

$$q(\varphi) = (1 - \sin\varphi)^{\frac{3}{4}}$$

One can verify that c_Q is only 3.1 % bigger than the factor c of the optimal strategy. The ratio $f_Q(\varphi)$ takes on its maximum value 1.25 at $\varphi = 0$. Similar approximations can be found for other values of β.

Although a formula for the optimal solution can not be given in a closed form, for the robot there is a simple method for finding that path, see Figure 9. This answers a question rised by S. Skyum. For the angle α between the tangent to the path at the actual position and the line from the actual position to the corner, the identity $\tan\alpha = \frac{r}{r'}$ holds. We can eliminate the derivative r' from Equation 4 and obtain the formula

$$\alpha = \arcsin\frac{r}{c_\beta \cos\varphi}$$

which means that the robot can, at each time, calculate its walking direction α if only the current distance to the corner r and the angle φ are known, as well as the constant c_β which can be computed in advance. In particular, the length of the path

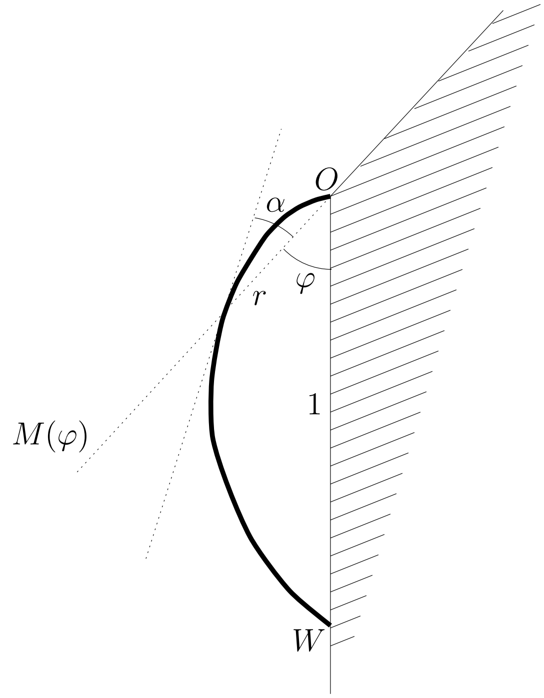

Figure 9: Tangent direction α can be computed on the way.

from the beginning to the actual position needs *not* to be known. For example, the initial angle for $\varphi = 0°$ and $r = 1$ is $\alpha = \arcsin \frac{1}{c_\beta}$. For $\beta > 0$, the robot follows that rule only until the actual direction is perpendicular to $M(\pi - \beta)$. At that point, it should go straight, as we have seen in Section 6.

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