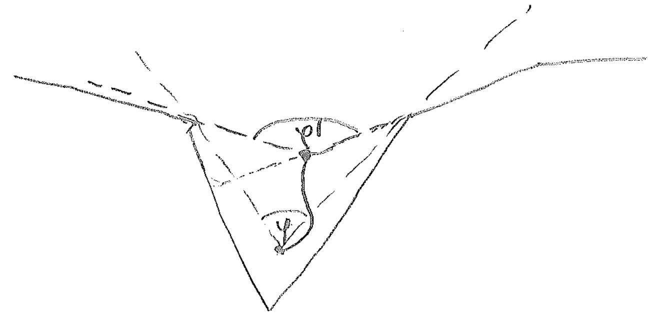
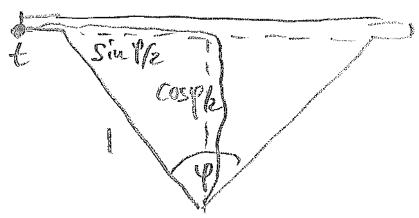
clear: while advancing through a funnel, the opening angle φ grows
$\Rightarrow$ use φ to parameterize strategy



Generalization of Lemma 3: $\quad K_\varphi := \sqrt{1+\sin\varphi}$

Lemma 4 With opening angle $\varphi \geq \frac{\pi}{2}$, no strategy can achieve a competitive factor $< K_\varphi$

Proof:



Each strategy can be forced to walk at least
$\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} = \sqrt{1+\sin\varphi} = K_\varphi$

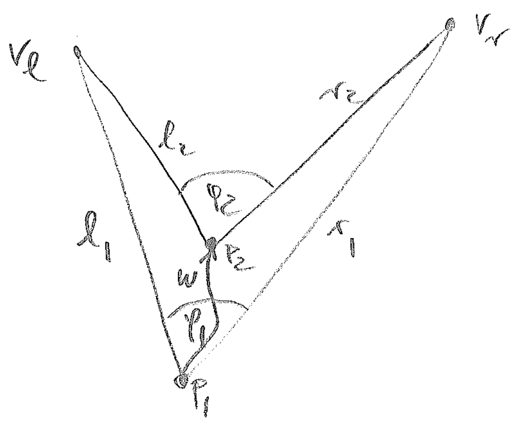
square

$\boxed{\text{Lemma 4}}$

Idea: Try to design strategy that achieves exactly these factors! How?

Backwards reasoning: $\varphi = \pi \Rightarrow$ robot can see both caves
$\Rightarrow$ can walk straight towards
$K_\pi = 1$ ✓

Let $\frac{\pi}{2} \leq \varphi_1 < \varphi_2 \leq \pi$, and assume that we already have strategy that works for all funnels with opening angle φ_2. How can we extend it to φ_1?



Assume, for a moment, that v_l, v_r don't change.

Let w be the length of robot's path from p_1 to p_2

if $t = v_\ell$: $|\text{Hotelpath}| \le K_{\varphi_2} \cdot l_2 + w \overset{!}{\le} K_{\varphi_1} \cdot l_1$

$t = v_r$: " $\le K_{\varphi_2} \cdot r_2 + w \overset{!}{\le} K_{\varphi_1} \cdot r_1$

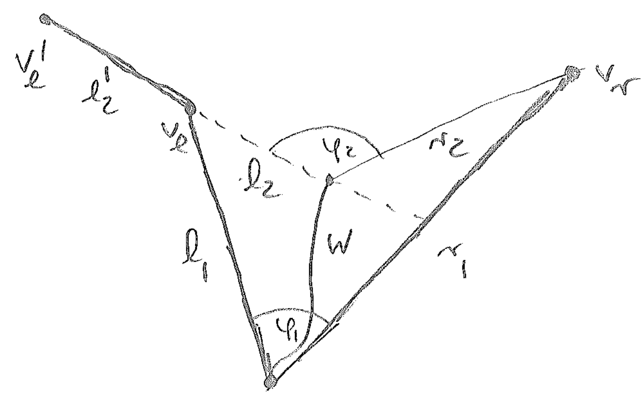
by assumption WANTED!

so, we want to guarantee

$$(1) \qquad w \overset{!}{\le} \min\left(K_{\varphi_1} \cdot r_1 - K_{\varphi_2} \cdot r_2 ,\; K_{\varphi_1} \cdot l_1 - K_{\varphi_2} \cdot l_2 \right)$$

Observation: Condition (1) is additive; if it holds for path W_{12} from φ_1 to φ_2, and for W_{23} from φ_2 to φ_3, then it holds for $W = W_{23} W_{12}$.

What would happen when v_ℓ changes to v_ℓ'?



$$w \le K_{\varphi_1} l_1 - K_{\varphi_2} l_2 = K_{\varphi_1} l_1 - K_{\varphi_2} l_2 + K_{\varphi_2} l_2' - K_{\varphi_2} l_2'$$

$$\le K_{\varphi_1}(l_1 + l_2') - K_{\varphi_2}(l_2 + l_2'),$$

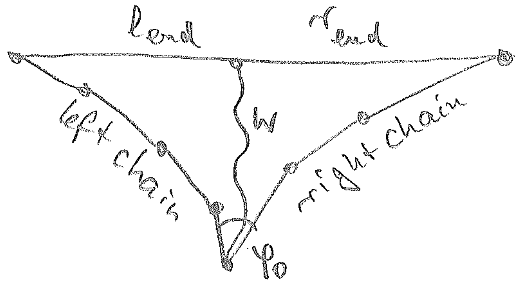
because $K_{\varphi_2} l_2' \le K_{\varphi_1} l_2'$ since $0 < \sin\varphi_2 \le \sin\varphi_1$.

Thus, if condition (1) can be satisfied for all small steps where v_ℓ, v_r do not change, we would obtain, by addition

$$W \le \min\left(K_{\varphi_0} \cdot \text{length of left chain} - K_\pi l_{\text{end}} ,\right.$$

$$\left. K_{\varphi_0} \cdot \text{length of right chain} - K_\pi r_{\text{end}} \right)$$

for each funnel of opening angle $\varphi_0 \geq \frac{\pi}{2}$

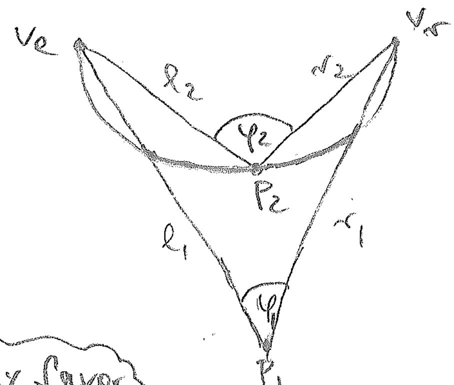


$\Longrightarrow$ robot's path $= W + \begin{Bmatrix} l_{end} \\ r_{end} \end{Bmatrix} \leq K_{\varphi_0} \cdot \begin{Bmatrix} |\text{left chain}| \\ |\text{right chain}| \end{Bmatrix}$

$K_{\pi} = 1$

remains to show: curve satisfying (1) can be constructed!

how?



all points having angle φ_2 lie on a circle through v_l, v_r.

Let's try to move p_2 to such a position where

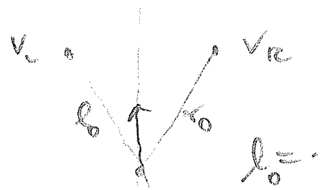
(intuition: why favor one side?)

$K_{\varphi_1} l_1 - K_{\varphi_2} l_2 = K_{\varphi_1} r_1 - K_{\varphi_2} r_2$ holds, i.e.,

$$(2) \qquad K_{\varphi_2}(l_2 - r_2) = K_{\varphi_1}(l_1 - r_1) = \underbrace{K_{\varphi_0}(l_0 - r_0)}_{\text{initial position}} =: A$$

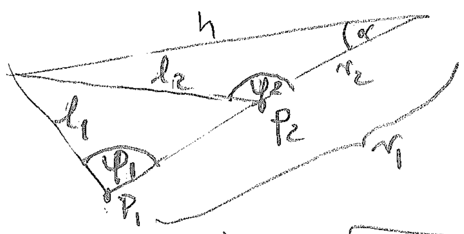
as long as v_l, v_r don't change

easy case: if $l_0 = r_0$: walk along bisector of v_l, v_r



Lemma 5 We can always find a point p_2 on the φ_2-circle spanned by v_l, v_r inside the triangle (p_1, v_r, v_l).

Proof Suppose p_2 is in rightmost position, $\in \overline{p_1 v_r}$.



$$\frac{h}{\sin\varphi_2} = \frac{l_2}{\sin\alpha}, \quad \frac{h}{\sin\varphi_1} = \frac{l_1}{\sin\alpha}$$

$$\Rightarrow \quad l_1 \sin\varphi_1 = l_2 \sin\varphi_2 \qquad (3)$$

function $\frac{K_\varphi}{\sin\varphi} = \sqrt{\frac{1}{\sin^2\varphi} + \frac{1}{\sin\varphi}}$ is increasing on $[\frac{\pi}{2}, \pi]$

$$\Rightarrow \quad \frac{K_{\varphi_1}}{\sin\varphi_1} \le \frac{K_{\varphi_2}}{\sin\varphi_2} \quad \underset{(3)}{\Longrightarrow} \quad K_{\varphi_1} l_1 \le K_{\varphi_2} l_2$$

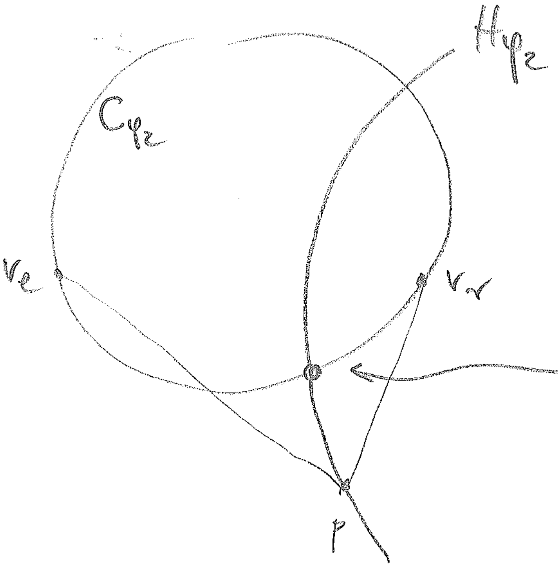
$$\Rightarrow \quad K_{\varphi_1}(l_1 - r_1) \underset{\substack{\uparrow \\ K_{\varphi_2} \le K_{\varphi_1} \\ r_2 \le r_1}}{\le} K_{\varphi_2}(l_2 - r_2).$$

For p in leftmost position, $K_{\varphi_1}(l_1 - r_1) \ge K_{\varphi_2}(l_2 - r_2)$ holds. By continuity, the claim follows.

$\boxed{\text{Lemma 5}}$

Condition (2): $\quad l_\varphi - r_\varphi = \frac{A}{K_\varphi}$

defines a hyperbola, $H_{\varphi_2} = \{p \in \mathbb{R}^2 \mid |pv_\ell| - |pv_r| = \frac{A}{K_{\varphi_2}}\}$



intersection point of C_{φ_2} and H_{φ_2}
$(X(\varphi_2), Y(\varphi_2))$ wanted!

One can show, by some calculations,

$$X(\varphi) = \frac{A}{2} \frac{\cot\frac{\varphi}{2}}{1+\sin\varphi} \sqrt{\left(1+\tan\frac{\varphi}{2}\right)^2 - A^2}$$

$$Y(\varphi) = \frac{1}{2} \cot\frac{\varphi}{2} \left(\frac{A^2}{1+\sin\varphi} - 1\right).$$

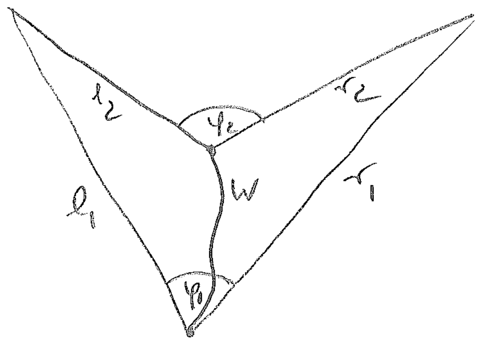
It remains to show that this curve really satisfies

(1) $\quad W \leq \min\left(K_{\varphi_1} r_1 - K_{\varphi_2} r_2 ,\ K_{\varphi_1} l_1 - K_{\varphi_2} l_2\right)$

where

$$W = \int_{\varphi_1}^{\varphi_2} \sqrt{X'(\varphi)^2 + Y'(\varphi)^2}\, d\varphi$$

arc length formula



that is, we need to show

$$\int_{\varphi_1}^{\varphi_2} \sqrt{X'(\varphi)^2 + Y'^2(\varphi)}\, d\varphi \leq K_{\varphi_1} l_1 - K_{\varphi_2} l_2 = \int_{\varphi_1}^{\varphi_2} \left(-K_\varphi l_\varphi\right)' d\varphi$$

It is sufficient to prove

$$\sqrt{X'(\varphi)^2 + Y'^2(\varphi)} \leq -\left(K_\varphi l_\varphi\right)' \quad \forall \varphi \in [\tfrac{\pi}{2}, \pi].$$

Can be done, but requires quite some work.

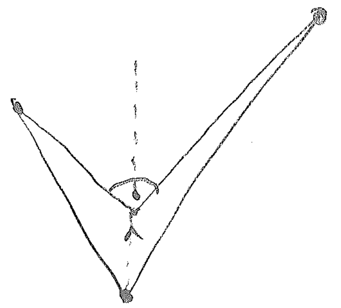
Worst factor: $\quad K_{\frac{\pi}{2}} = \sqrt{\sin\frac{\pi}{2} + 1} = \sqrt{2}$

for $\varphi \in [\frac{\pi}{2}, \pi]$

(9)

for starting angles $\varphi \in [0, \frac{\pi}{2}]$: follow angular bisector until $\varphi = \frac{\pi}{2}$;

then follow $(X(\varphi), Y(\varphi))$!

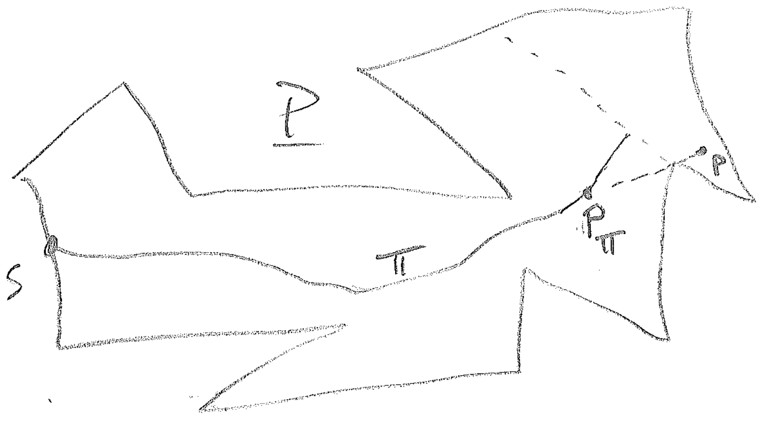


<u>Theorem</u> This strategy is competitive with factor $\sqrt{2}$, hence optimal. $\boxed{\text{Theorem}}$

Back to searching in general polygons!

We know: no constant factor possible as compared to shortest path.

New model: best search path



at point $p_\pi \in \pi$, point p becomes visible for the first time

<u>Def:</u> (i) For each exploration path (not tour!) in P from s, let

$$sr(\pi) := \sup_{p \in P} \frac{|\pi_s^{p_\pi}| + |p_\pi p|}{sp(s,p)}$$

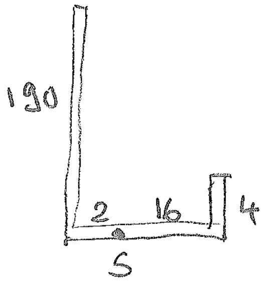
where $sp(s,p)$ denotes the shortest s-to-p path in P.

(ii) A search path π_{opt} that minimizes $sr(\pi)$ is called an optimal search path for P.

Let $sr_{opt} := \min_{\pi} sr(\pi) = sr(\pi_{opt})$.

Optimal search paths are hard to compute (in graphs: NP-hard, Koutsopias, Papadimitriou, Yannakakis '96)

<u>Example</u>



Should π_{opt} explore left or right leg first?

left first:

$$sr = \max\left(\frac{2+190}{2+190}, \frac{4+16+4}{16+4}\right) = 1+\frac{1}{5}$$

right first:

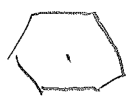
$$sr = \max\left(\frac{16+4}{16+4}, \frac{32+2+190}{2+190}\right) = 1+\frac{1}{6}.$$

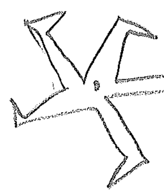
<u>Def:</u> Search strategy S is search-competitive with factor C $:\Longleftrightarrow$ $sr(\pi_S) \le C\cdot sr(\pi_{opt}) + B$ holds, for some constant B and all simple polygons.

<u>Remarks</u> (i) We are comparing the quality of search pat

(ii) Some polygons can be easier searched than others

This is reflected in sr_{opt}!





Its value can be arbitrarily large, depending on P.

(iii) Suppose S is search-competitive with factor C, as in (ii).

Then, S is $C\cdot sr_{opt}$ - competitive in the standard sense.

(99)

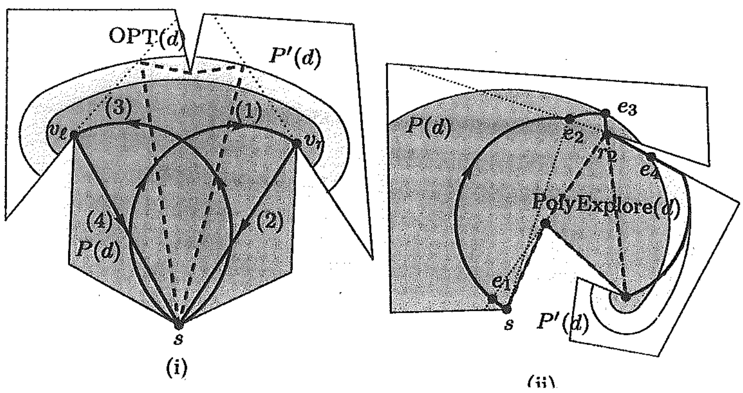
<u>Theorem</u> There is a search-competitive strategy with factor 213 for simple polygons.

Proof: <u>Idea:</u> Combine Doubling and PolyExplore (

Given simple polygon P, let $P_d := \{z \in P \mid |sp(s,z)| \leq d$
and start point s

We can run PolyExplore(d), exploring only left/right vertices of P_d; both robot's path and $W_{opt}(P_d)$ are allowed to leave P_d an run through P:

...OF THE OPTIMAL SEARCH RATIO 893



Let π_{opt} be an optimal search path for P, and $last_d \in \pi_{opt}$ the point on π_{opt} from which the last point of P_d becomes visible.

<u>Clear:</u> moving from s forth and back on $(\pi_{opt})_s^{last_d}$ explores polygon P_d

$$\Rightarrow \quad |W_{opt}(d)| \leq 2 \left|(\pi_{opt})_s^{last_d}\right|$$

Moreover,

$$sr_{opt} \geq \frac{(\pi_{opt})_s^{last_d}}{d} \geq \frac{W_{opt}(d)}{2d} \qquad (*)$$

by definition of sr_{opt}

Now we run ExplorePolygon(d) for $d = 2^i$, $i = 1, 2, \ldots$
Each run terminates at s. Assume point t was

just missed for $d = 2^j$ $\Rightarrow$ robot now:

- performs ExplorePolygon(2^{j+1}) to reach point q that sees
- completes this exploration at s
- walks from s to t on shortest path sp(s,t)

$$\Rightarrow \quad \frac{\text{robot's path to t}}{\text{shortest path to t}} \le \frac{\sum_{i=1}^{j+1} |\text{ExplorePolygon}(2^i)| + |sp(s,t)|}{|sp(s,t)|}$$

$$\underset{\text{Theorem B}}{\le} \quad 1 + \frac{26.5 \cdot \sum_{i=1}^{j+1} W_{opt}(2^i)}{|sp(s,t)|}$$

$$\underset{(*)}{\le} \quad 1 + 26.5 \, \frac{\overbrace{\sum_{i=1}^{j+1} 2^{i+1}}^{< 2^{j+3} = 8 \cdot 2}}{\underbrace{|sp(s,t)|}_{\ge 2^j \text{, otherwise t would have been disco… in previous exploration}}} \, sr_{opt}$$

$$< (1 + 8 \cdot 26{,}5) \, sr_{opt} = 213 \cdot sr$$

$\boxed{\text{Theorem}}$

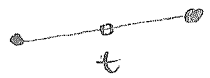
Theorem shows: (unknown) optimal search path can be approximated up to a constant factor — even on-line.

("if the keys get lost, it doesn't matter if it happens at home or somewhere else") (?)

The same approach works for searching undirected graphs (of unit edge length). One can use Doubling and the strategy for tethered robots.

(0) from a vertex, robot sees edges but not their lengths.

Model: (i) target t can be located on an edge



$\Rightarrow$ π_{opt} has to visit all edges

(ii) graph is planar, i.e., can be drawn in the plane without crossings.

(iii) no loops  or multiple edges 

(iv) edges undirected.

(ii), (iii) $\underset{\substack{\text{Euler} \\ v-e+f=c+1}}{\Longrightarrow}$ $e \in O(v)$. That is, the number of edges in the theorem for exploration with tethered robots can be replaced by number of vertices, times a constant.

Theorem There is a search-competitive strategy with constant factor for searching above graphs.

good exercise: compute constant for graphs with unit length edges

Trick in last two theorems:
Use efficient exploration strategy that can be restricted to subproblems of bounded size; double the size.

Graph searching is much harder when t can only be placed at vertices.
Here, π_{opt} need only visit vertices!

Euler's formula Let $G=(V,E)$ be a plane graph (i.e., a geometric drawing in $\mathbb{R}^2$ without crossings).

Let $v = \#$ vertices, $e = \#$ edges, $f = \#$ faces, $c = \#$ connected components

Then, $v - e + f = c + 1$.

Proof: By induction on e.

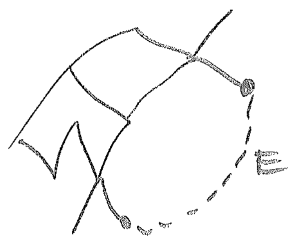
$e = 0$: v vertices, $c = v$, $e = 0$, $f = 1$

$e \to e+1$ by inserting an edge:

case 1 : two connected components joined by new edge E

$c \leftarrow c-1$
$e \leftarrow e+1$

case 2 : two vertices of same component connected by E



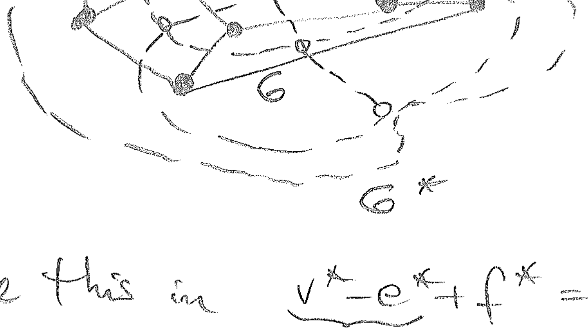
$f \leftarrow f+1$
$e \leftarrow e+1$

[Euler]

Consequence Any plane drawing of a planar graph has the same number of faces.

Now, assume G does not contain  or 

$\Rightarrow$ in the dual graph G^*: each vertex of degree $\geq$ [illegible]



faces of G $\leftrightarrow$ vertices of G^*
edges of G $\leftrightarrow$ edges of G^*

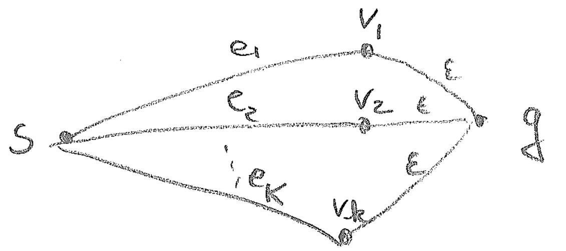
$\Rightarrow 3v^* \leq 2e^*$

substitute this in $\underbrace{v^* - e^*}_{\leq -\frac{1}{3}e^*} + f^* = c^* + 1$

$$\Rightarrow \quad e = e^* \leq 3(f^* - c^* - 1) < 3f^* - 6 = 3v - 6. \qquad (101.2$$

Lemma There is no constant-factor search-competitive strategy for vertex search in planar graphs.

Proof



Let robot run for a while, making edges larger, as necessary

At any given time, let $d_1 \geq d_2 \geq \dots \geq d_k \geq 0$ be the depths to which robot has visited $e_1, \dots, e_k$.

when $d_k \geq k^2$: set all edge lengths to $d_i + \varepsilon$

$$\Rightarrow \quad |\pi_{opt}| = (d_k + \varepsilon) + \varepsilon + (k-2)2\varepsilon + \varepsilon = d_k + (2k-1)\varepsilon$$

all shortest s–p paths of length $\geq d_k + \varepsilon$

$$\Rightarrow \quad sr(\pi_{opt}) \leq \frac{d_k + (2k-1)\varepsilon}{d_k + \varepsilon}$$

<u>case 1</u> $d_k \leq 1 \Rightarrow \dfrac{d_k + (2k-1)\varepsilon}{d_k + \varepsilon} \leq 2k-1 < 2k$

$\dfrac{a+b}{a+c} \leq \dfrac{b}{c}$ if $a, b, c > 0$ and $b > c$

but $|\pi_s^{v_k}| \geq k^2$ and shortest path $sp(s, v_k) = d_k + \varepsilon$

$$\Rightarrow \quad sr(\pi) \geq \frac{k^2}{d_k + \varepsilon} \underset{d_k \leq 1}{\geq} \frac{k^2}{1+\varepsilon} > \frac{k}{4} \cdot 2k > \frac{k}{4} \cdot sr(\pi_{opt})$$

<u>case 2</u> $d_k \geq 1 \Rightarrow \dfrac{d_k + (2k-1)\varepsilon}{d_k + \varepsilon} \leq 1 + \delta$ for any δ, if ε small enough

let v_i be first v-vertex visited by strategy

$$\Rightarrow \quad |\Pi_s^{v_i}| \geq d_1 + d_2 + \dots + d_k \geq k\, d_k$$

$$sp(s, v_i) \leq d_k + 3\varepsilon$$

$$\Rightarrow \quad sr(\Pi) \geq \frac{k\, d_k}{d_k + 3\varepsilon} > \frac{k}{2}(1+\delta) \geq \frac{k}{2}\, sr(\Pi_{opt})$$

Thus, there is no constant bound. $\boxed{\text{Lemma}}$

<u>A classic</u>: Searching for a line at known location

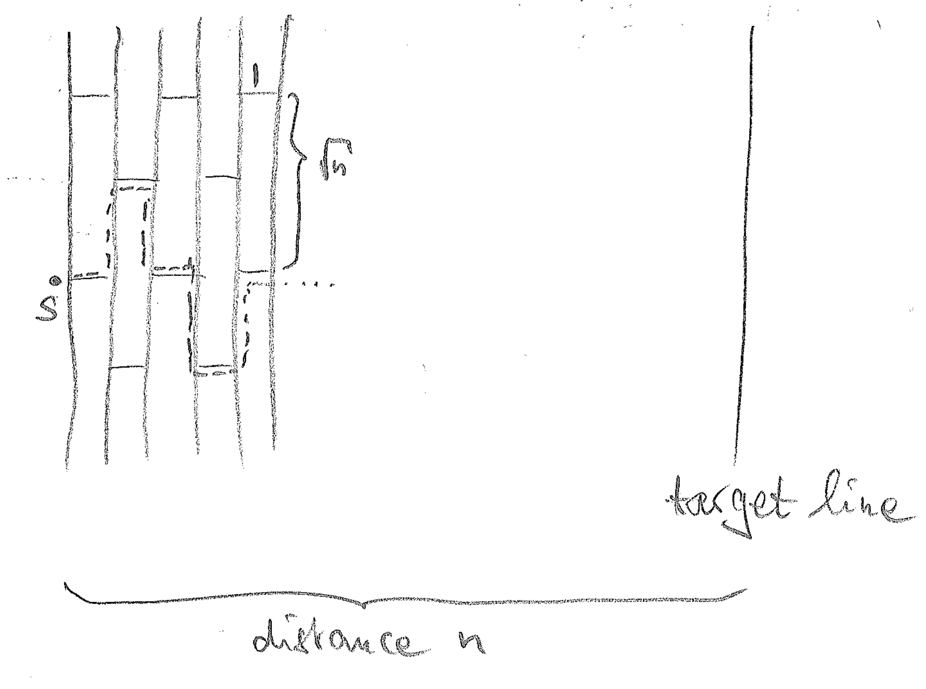
n obstacles: $\boxed{}\, b$, a $a, b \geq 1$

robot: tactile sensor (no vision), knows line position

Lowerbound (Papadimitriou, Yanakakis '89):

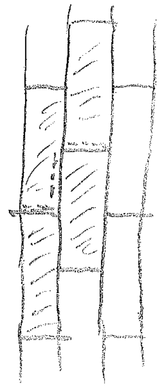
<u>Theorem</u> Every deterministic search algorithm for such a scene can be forced to produce a path at least $c \cdot \sqrt{n}$ times longer than the shortest path.

<u>Proof</u> Imagine robot starts in front of a brick wall



robot's path contains $v \geq n$ vertical segments of length $\frac{\sqrt{n}}{2}$ each

at each vertical segment, at most 4 bricks are touched by robot's path:



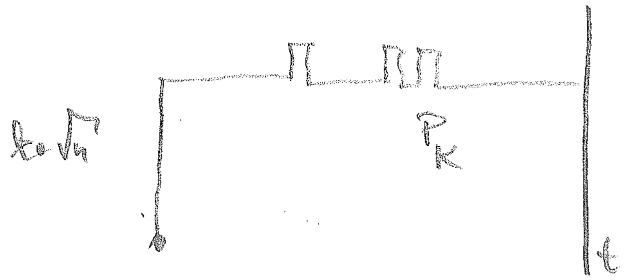
$\Rightarrow$ $\leq 4v$ bricks touched altogether

Remove all bricks not touched by robot $\longrightarrow$ environment

robot's path π in E same as before (deterministic!)

$$|\pi| \geq v \cdot \frac{\sqrt{n}}{2}$$ (counting only vertical segments)

shortest path in E: consider paths P_k



$\longleftarrow$ many bricks missing

for $k = 1, 2, \ldots, \sqrt{n}$. Clear: $k \neq l \Rightarrow P_k, P_l$ do not touch the same bricks

$\Rightarrow$ for some k, P_k touches $\leq \frac{4v}{\sqrt{n}}$ bricks

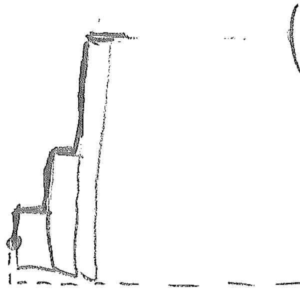
$$\Rightarrow |SP(s,t)| \leq |P_k| \leq k\sqrt{n} + \underbrace{n}_{\text{horizontal distance}} + \underbrace{\frac{4v}{\sqrt{n}}\sqrt{n}}_{\text{vertical segments}}$$

$$\leq 2n + 4V \leq 6v$$

Theorem

Can this bound be achieved?

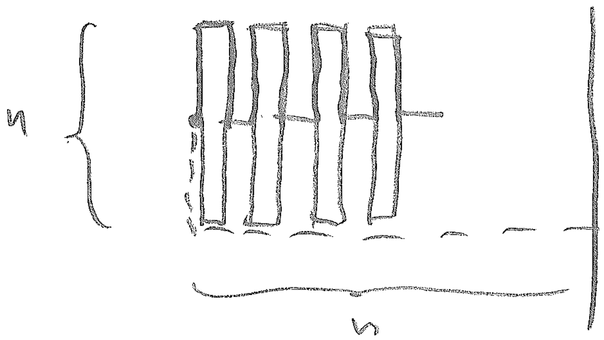
<u>1st attempt</u>: always walk to closest corner; continue from there



$|\text{path}| > 2^n$

$|\text{shortest path}| = n+1$

<u>2nd attempt</u>: always walk to closest corner and return



$|\text{path}| \geq (n-1)\,n$

$|\text{shortest path}| \leq \frac{3}{2}n$