

Selecting 2 points of $N(B)_{mid}$ closest to

σ_B in cone k

proportional to $|N(B)_{mid} \cap k|$ (Lemma 26)

(all v.) (Greedy to 2 smallest (linear), run through

the corresponding set

$O(\ln)$ sets of size $O(k)$: $N(\frac{1}{2}B)$

$\rightarrow (2-1)$ smaller
over

Size of all sets $L(a) \in O(\ln k)$

Selecting 2 points of $L(a)$

proportional to $|L(a)|$ (Lemma 26)

(all v.)

$O(n \log n + n \cdot k)$

□

Another application: Spanner construction:

Theorem 31 A $(1-\epsilon)$ -spanner for a $(\epsilon > 0, \text{fixed})$

Set S of n points in \mathbb{R}^d can be
computed in $O(\frac{1}{\epsilon^d} n + d n \log n)$ time

(size $O(\frac{1}{\epsilon^d} n)$ edges)

Proof:

$$t = \frac{8+4}{5-4} = (1+\epsilon) \quad \text{Lemma 21}$$

$$\Leftrightarrow s = \left(\frac{8}{\epsilon} + 4\right)$$

□

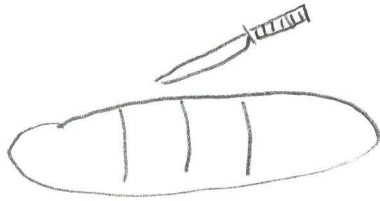
Brunn's Inequality

Matousek

Chapter 12.2

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Problem: Bread slices volume from a bread V .

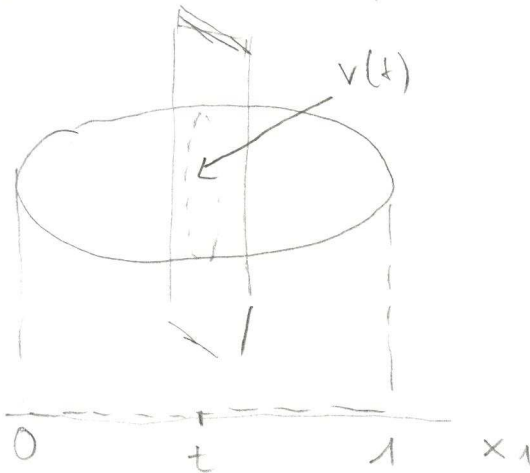


Take the middle one
because it is near the
smallest V .

This holds in any dimension V .

Situation:

$C \in \mathbb{R}^{d+1}$ convex, $[t_{\min}, t_{\max}] = [0, 1]$
w.o.o.g.



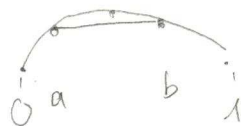
Projection of C onto x_1 -axis

$v(t) := \text{Volume of } C \cap \{\text{Hyperplane } x_1 = t\}$

Theorem 32: (Brunn's inequality) \Leftarrow

The function $v(t)^{\frac{1}{d}}$ is concave over $[0, 1]$.

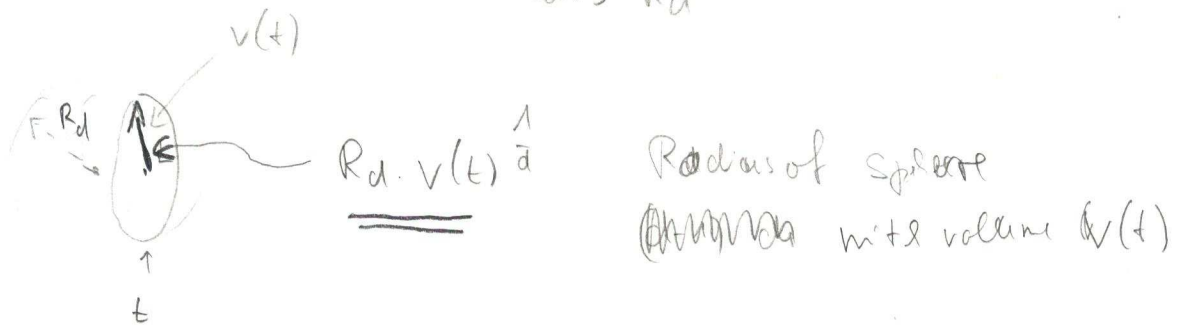
$f(t)$ concave over $[0, 1]$



$\Leftarrow \Leftrightarrow \forall a, b \in [0, 1] : (1-t)f(a) + tf(b) \leq f(t) \quad \forall t \in [a, b]$

$$V(t)^{\frac{1}{d}} \quad ?$$

Intuition: d -dimensional sphere of volume 1
with radius R_d

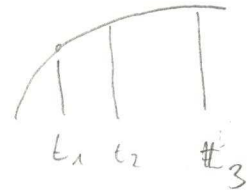


Brunn's inequality implies Brunn - solution:

$$V(t)^{\frac{1}{d}} \text{ concave } \forall 0 \leq t_1 \leq t_2 \leq t_3 \leq 1$$

$$V(t_2)^{\frac{1}{d}} \geq \min \left(V(t_1)^{\frac{1}{d}}, V(t_3)^{\frac{1}{d}} \right)$$

$$\Rightarrow V(t_2) \geq \min \left(V(t_1), V(t_3) \right)$$



[$V(t)$ is not concave in general]
 $d \geq 3$

[limit points
(compact) } inside]

Proof of Theorem 32 makes use of

Theorem 33 (Brunn - Minkowski - Inequality)

Let $A, B \subseteq \mathbb{R}^d$ compact non-empty:

$$\forall \text{gen } \text{vol}(A)^{\frac{1}{d}} + \text{vol}(B)^{\frac{1}{d}} \leq \text{vol}(A+B)^{\frac{1}{d}}$$

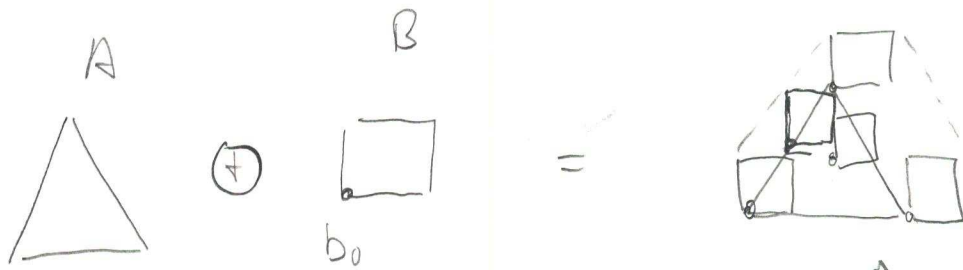
\Leftrightarrow closed and
bounded
[ball of finite
radius]

$$A \oplus B := \{a+b \mid a \in A, b \in B\}$$

A' translation of A \Rightarrow $A' \oplus B'$ translation of $A \oplus B$
 B' translation of B

Position of $A \oplus B$ depends on the choice of the coordinate system. (values independent from translations)

Interpretation:



Fix A , pick b_0 in B , translate B into all situations where b_0 is in A
 (Result modulo translation)

Proof of Theorem 32 with Theorem 33:

Let $A, B \in \mathbb{R}^d$ be convex. For $t \in [0, 1]$

Consider $Z_t := (1-t)A \oplus tB = \{(1-t)a + tb \mid a \in A, b \in B\}$

"Morphing of A into B " (A, B will be the ^{cut} cuts)

Realize Morphing geometrically

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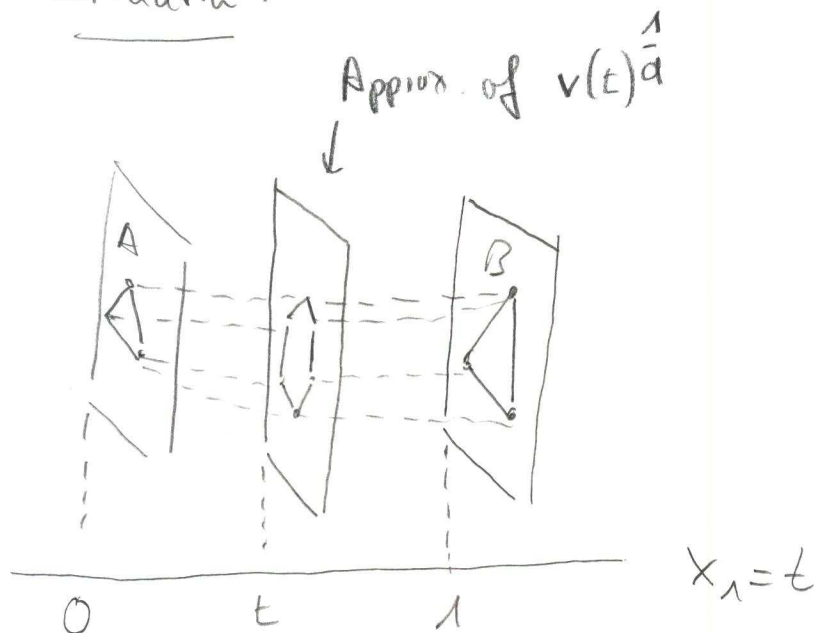
Place A and B into \mathbb{R}^{d+1}

A into hyperplane $\{x_1=0\}$

B into hyperplane $\{x_1=1\}$

Line segments between them!

Situation:



Lemma 34: A, B convex

$$\text{conv} \left(\left(\sum_{0} \times A \right) \cup \left(\sum_{1} \times B \right) \right) \stackrel{!}{=} \bigcup_{t \in [0,1]} \underbrace{\sum_{t} \times ((1-t)A + tB)}_{2t} =: C'$$

Proof: " \supseteq " Clearly C' contains
 $\sum_{0} \times A$ and $\sum_{1} \times B$ and
all lines from A to B

Any convex set that contains $\sum_{0} \times A$ and $\sum_{1} \times B$
also contains C'

" \subseteq " clearly: $\{0\} \times A$ and $\{1\} \times B \subseteq C'$
 for $\lambda = 1, 0$.

C' is convex $(\Rightarrow \text{conv}(\dots) \subseteq C')$ $\bigcap_{C' \subseteq A}$

$$p = (t_1, (1-t_1)a_1 + t_1 b_1) \quad q = (t_2, (1-t_2)a_2 + t_2 b_2) \in C'$$

$(a_1, a_2 \in A, b_1, b_2 \in B)$

Line segment between p and q :

$$r \in [0, 1] \quad (1-r)p + r q =$$

$$\left((1-r)t_1 + r t_2, (1-r)((1-t_1)a_1 + t_1 b_1) + r((1-t_2)a_2 + t_2 b_2) \right)$$

$$= \left(\underbrace{(1-r)t_1 + r t_2}_{\delta}, \underbrace{(1-r)((1-t_1)a_1 + t_1 b_1) + r((1-t_2)a_2 + t_2 b_2)}_{\in A(\text{conv})} \right)$$

A, B convex

$$+ \left(\underbrace{(1-r)t_1 + r t_2}_{\delta}, \underbrace{(1-M)b_1 + M b_2}_{\in B(\text{conv})} \right) \in C'$$

where $\lambda = \frac{r(1-t_2)}{1-\delta}$

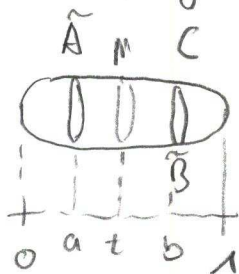
$\mu = \frac{r t_2}{\delta}$ (verify!)
 $\in [t_1, t_2]$

(Simple computation)

□

New Proof of Theorem 32:

Let $C \in \mathbb{R}^{d+1}$ convex with projection $[0,1]$
onto x_1 -axis.



Let A and B

be two "cuts"

$$\text{Let } \tilde{A} = C \cap \{x_1 = a\}, \quad \tilde{B} = C \cap \{x_1 = b\} \quad \tilde{A}, \tilde{B} \in \mathbb{R}^{d+1}$$

Volumes (d-dim) $v(a), v(b)$

Show: $t \in [a, b]$ ($v(t)^{\frac{1}{d}}$ is concave)

$$(1-t) v(a)^{\frac{1}{d}} + t v(b)^{\frac{1}{d}} \leq v(t)^{\frac{1}{d}}$$

$\text{vol}(\tilde{A}) \qquad \qquad \text{vol}(\tilde{B}) \qquad \qquad \text{vol}(M)$

w.p.o.g. $a=0, b=1$

Instead of C and M consider $C' := \text{conv}(\tilde{A} \cup \tilde{B}), M' := C' \cap \{x_1 = t\}$

Clearly, $C' \subseteq C, M' \subseteq M$ (C convex)

(Lemma 34) $\Rightarrow M' = (1-t)\tilde{A} + t\tilde{B}$ (volume is d-dim.)

$$\begin{aligned} \Rightarrow (1-t) \text{vol}(\tilde{A})^{\frac{1}{d}} + t \text{vol}(\tilde{B})^{\frac{1}{d}} &= \overset{\text{conv. (scaling)}}{\text{vol}((1-t)\tilde{A})^{\frac{1}{d}} + \text{vol}(t\tilde{B})^{\frac{1}{d}}} \\ &\leq \text{vol}((1-t)\tilde{A} + t\tilde{B})^{\frac{1}{d}} = \text{vol}(M')^{\frac{1}{d}} \\ &\stackrel{\text{Theorem 33}}{\leq} \text{vol}(M)^{\frac{1}{d}} \\ &\stackrel{\uparrow}{\leq} \text{vol}(M)^{\frac{1}{d}} \quad \square \end{aligned}$$

Now the more general result:

Proof: (Brunn-Minkowski)

First A, B union of finitely many closed axis-parallel boxes ("bricks") with disjoint interior

Show the proof for the union of $\underline{2}$ such boxes (Afterwards arbitrary A, B approximated)

Ind. case

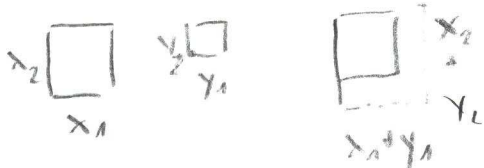
$k=2$ (non-empty) A and B are bricks and $A \oplus B$ is a brick also

A given by side length x_1, x_2, \dots, x_d

B given by " " y_1, y_2, \dots, y_d

Show: $\left(\prod_{i=1}^d x_i\right)^{1/d} + \left(\prod_{i=1}^d y_i\right)^{1/d} \leq \left(\prod_{i=1}^d (x_i + y_i)\right)^{1/d}$

Holds: example $d=2$



$$\sqrt{x_1 x_2} + \sqrt{y_1 y_2} \leq \sqrt{(x_1 + y_1)(x_2 + y_2)} \quad (*)^2$$

$$\Leftrightarrow \left[(\sqrt{x_1} \sqrt{x_2} + \sqrt{y_1} \sqrt{y_2})^2 \right] \leq$$

$$\cancel{x_1 x_2} + \cancel{y_1 y_2} + 2\sqrt{(x_1 y_1)(y_1 y_2)} \quad (x_1 + y_1)(x_2 + y_2)$$

$$\left[\begin{aligned} 2\sqrt{AB} &\leq AB \\ 4AB &\leq (A+B)^2 \\ &= A^2 + 2AB + B^2 \\ 0 &\leq (A-B)^2 \end{aligned} \right] \Leftrightarrow \begin{aligned} &\leq \cancel{x_1 x_2} + \cancel{y_1 y_2} \\ &+ y_1 x_2 + y_2 x_1 \end{aligned}$$

$$d \geq 2 \quad \alpha = 2$$

Lemma 35 $x_1, \dots, x_d, y_1, \dots, y_d \geq 0$

$$\text{Then } \left(\prod_{i=1}^d x_i \right)^{\frac{1}{d}} + \left(\prod_{i=1}^d y_i \right)^{\frac{1}{d}} \leq \left(\prod_{i=1}^d (x_i + y_i) \right)^{\frac{1}{d}}$$

Proof: Let $x_i^{\frac{1}{d}} =: v_i, y_i^{\frac{1}{d}} =: w_i$ and apply $(\)^d$

$$\left(\prod_{i=1}^d v_i + \prod_{i=1}^d w_i \right)^d \leq \prod_{i=1}^d (v_i^d + w_i^d)$$

$$\left[(x_1 x_2)^{\frac{1}{d}} = x_1^{\frac{1}{d}} x_2^{\frac{1}{d}} \right]$$

Binomial-coefficients

$$\Leftrightarrow \sum_{m=0}^d \binom{d}{m} (v_1 \dots v_d)^m (w_1 \dots w_d)^{d-m} \leq \sum_{m=0}^d \binom{d}{m} \underbrace{v_{i_1} \dots v_{i_m} w_{j_1} \dots w_{j_{d-m}}}_{\substack{\text{Binomial-coefficients} \\ \binom{d}{m} \text{ subterms} \\ i_1, \dots, i_m \in \{1, \dots, d\} \quad i_m \neq i_{m-1} \\ j_1, \dots, j_{d-m} \in \{1, \dots, d\} \quad j_m \neq j_{m-1}}}$$

Show: $0 \leq m \leq d \quad \frac{\binom{d}{m}}{\binom{d}{m}} \geq \frac{\binom{d}{m}}{\binom{d}{m}} \quad \left[\binom{d}{m} = \frac{d!}{m!(d-m)!} \right]$

$\lceil M \text{ fs} \rceil$

$$\frac{\lceil M \rceil}{\binom{d}{M}} = \frac{1}{\binom{d}{M}} \sum_{\substack{\{i_1, \dots, i_m\} \subseteq \{1, \dots, d\} \\ i_m \neq i_n \\ \{j_1, \dots, j_{d-m}\} \subseteq \{1, \dots, d\} \\ j_m \neq j_n}} v_{i_1}^d \dots v_{i_m}^d w_{j_1}^d \dots w_{j_{d-m}}^d$$

$\binom{d}{M}$
Summands

$$\geq \left(\prod_{\substack{\{i_1, \dots, i_m\} \subseteq \{1, \dots, d\} \\ i_m \neq i_n \\ \{j_1, \dots, j_{d-m}\} \subseteq \{1, \dots, d\} \\ j_m \neq j_n}} v_{i_1}^d \dots v_{i_m}^d w_{j_1}^d \dots w_{j_{d-m}}^d \right) \frac{1}{\binom{d}{M}}$$

with \geq from

$$\sqrt[n]{x_1 + \dots + x_n} \geq (x_1 x_2 \dots x_n)^{\frac{1}{n}}$$

$$\frac{d \binom{d-1}{m-1}}{\binom{d}{m}} = \frac{d! / (m-1)! (d-m)!}{d! / (m-1)! (d-m)!} = m$$

$$= \frac{dM}{\binom{d}{M}}$$

each v_i

occurs

$$\binom{d-1}{m-1} \text{ times!}$$

$\binom{d-1}{m-1}$ subsets of $\{1, \dots, d\} \setminus \{i\}$
can be enlarged by v_i

and w_j occurs $\binom{d-1}{d-m-1}$ times

$\binom{d-1}{d-m-1}$ subsets of $\{1, \dots, d\} \setminus \{j\}$
can be enlarged by w_j

$$\Rightarrow v_i^d \binom{d-1}{m-1} \frac{1}{\binom{d}{M}} = v_i$$

$$w_j^d \binom{d-1}{d-m-1} \frac{1}{\binom{d}{M}} = w_j^{d-M} \rightsquigarrow (v_1 \dots v_d)^M (w_1 \dots w_d)^{d-M}$$

□