WSPD - Pairs reported:

$(\{0\}, \{4,5\}), (\{0\}, \{7\}), (\{0\}, \{12,13,14,16\}), (\{4\}, \{5\})$
$(\{4,5\}, \{7\}), (\{4,5,7\}, \{12,13,14\}), (\{4,5,7\}, \{16\}), (\{12\}, \{13\})$
$(\{12,14\}), (\{12,13\}, \{16\}), (\{13\}, \{14\}), (\{14\}, \{16\})$

Some sets occur more than once $\quad m = 12$

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② **Analysis:** $m \in O(s^d d^{\frac{d}{2}} n)$ and Comp. time also.

(Takes some time!)

Q: How many pair (a, b_i) can occur?

$S_{b_i} \cap S_{b_j} = \phi \qquad$ (fixed a!)

Suppose (a,b) was reported $\quad S_a, S_b$ well separated

$\Rightarrow S_{\pi(a)}, S_b$ or $S_a, S_{\pi(b)}$ are NOT reported.

$\pi(a) =$ parent of a

Consider $S_{\pi(a)}, S_b \qquad$ Test FindPairs (a,b) implies:

$$L_{max}(R(b)) \le L_{max}\, R(\pi(a)) \qquad (*)$$

FindPairs(a,b)

(Split at "greater" set)

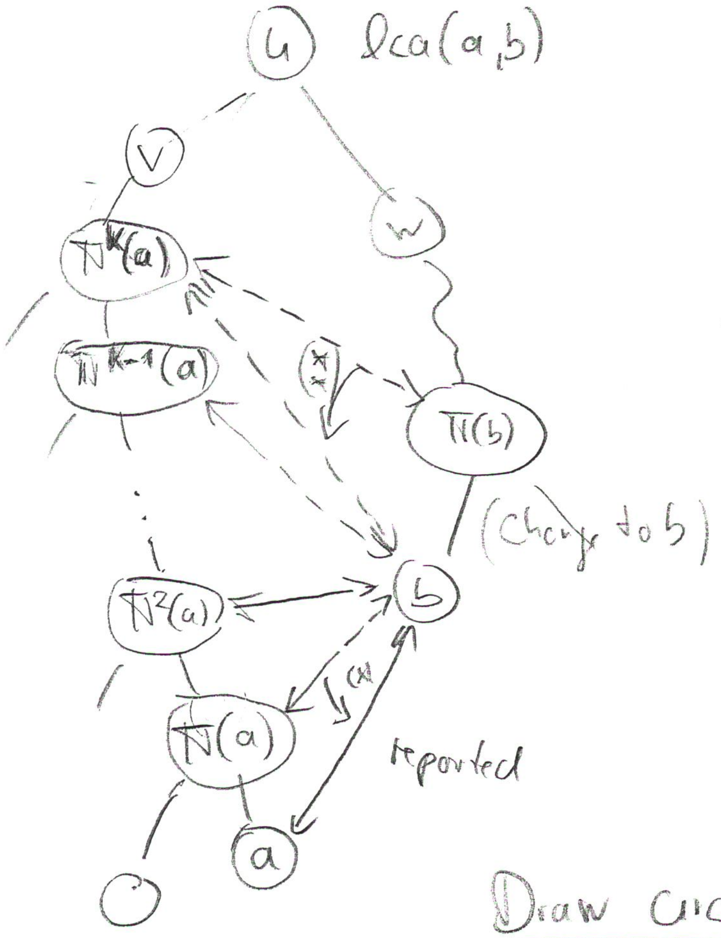
$$L_{max}(R(\pi(a))) \leq L_{max}(R(\pi^2(a)))$$ parents are always greater

$$\leq L_{max}(R(\pi(b))) \quad (**)$$

(Find Pairs)

(Charge to b)

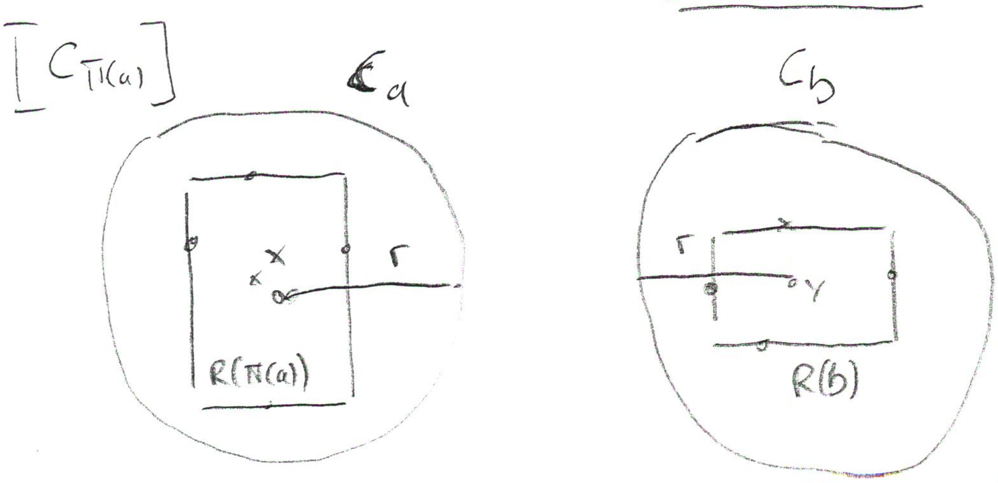
Situation:



$\pi(b) = u$

$\Rightarrow$ (**) trivially holds

Draw circles:



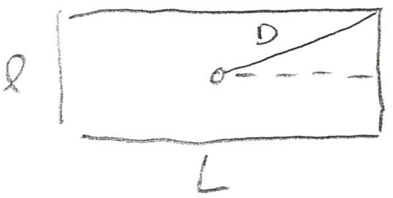
Circles C_a, C_b of radius $\frac{\sqrt{d}}{2} L_{max}(R(\pi(a))) := r$

around centers x of $R(\pi(a))$ and y of $R(b)$

- $R(b)$ lies in C_b because of (*)
- $R(\overline{N}(a))$ lies in C_a because of $\frac{\sqrt{d}}{2}$ factor

$\frac{\sqrt{d}}{2}$ length from center of d-dim box to furthest vertex, length of largest edge

$d=2$ $\qquad D \leq \frac{\sqrt{2}}{2} L$



$$D = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{l}{2}\right)^2} \leq \frac{1}{2}\sqrt{2L^2} = \frac{\sqrt{2}}{2} L$$

Two cases for the position of C_a and C_b
(circles x, y of $R(\overline{N}(a))$ and $R(b)$)

1. $C_a \cap C_b = \emptyset$

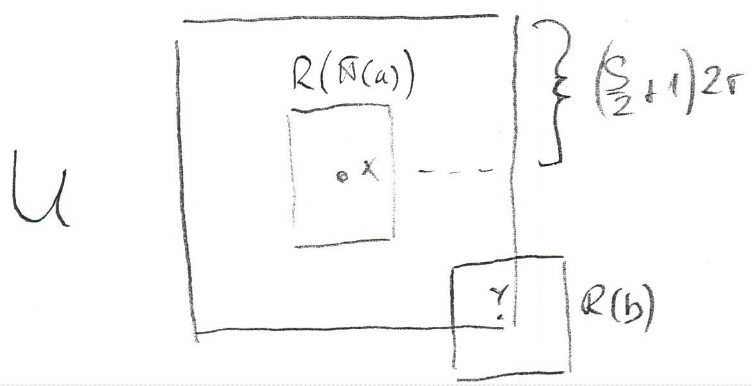
$$\Rightarrow |C_a C_b| < s \cdot r \Rightarrow |xy| = |C_a C_b| + 2r < \left(\frac{s}{2} + 1\right) 2r$$

not well-separated

2. $C_a \cap C_b \neq \emptyset \Rightarrow |xy| \leq 2r < \left(\frac{s}{2}+1\right) 2r$

boxes $R(\overline{N}(a))$ and $R(b)$ are close

$\Rightarrow y$ is contained in hypercube U of size $2\left(\frac{s}{2}+1\right)$ centered at x



Now assume that pairs $(a,b_1),(a,b_2),\ldots,(a,b_k)$ of this type were reported

$\Longrightarrow$ for $i \neq j$ S_{b_i}, S_{b_j} disjoint

uniqueness of WSPD

separated by hyperplanes

$\Longrightarrow$ $R(b_i), R(b_j)$ disjoint

Want to get a bound on k by a packing argument.

\# of pairs

Q: How many $R(b)$ fit into U?

Need to bound $R(b_i)$!

Lower bound of $R(b_i)$

Recap: Analysis $(a,b_1),(a,b_2),\ldots,(a,b_k)$

$\Longrightarrow$ $S_{\pi(a)}, S_{b_i}$ not well-separated

S_{b_i}, S_{b_j} disjoint separated by hyperplanes

Size of k?

In the proof (a slightly bigger

For the proof: (a slightly bigger container)

Construct Container $R_0(a)$ for bounding box $R(a)$

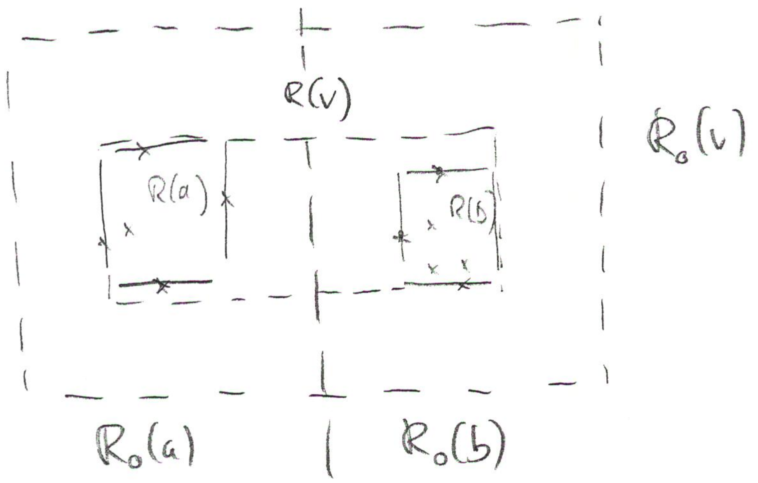
of set S_a of node a

Start with $R_0(s)$ · Hyper cube that contain $R(s)$

Split of the bounding box $R(v)$

use the same splitting hyperplane for $R_0(v)$

Containers do not shrink to fit point set



Clear: $S_v \cap S_w = \emptyset \Rightarrow R_0(v) \cap R_0(w) = \emptyset$

Lemma 21: $b \neq \text{root} \Rightarrow \underset{\substack{\uparrow \\ \text{smallest dimension}}}{L_{min}}(R_0(b)) \geq \frac{1}{2} \underset{\substack{\uparrow \\ \text{largest dimension}}}{L_{max}}(R(\pi(b)))$

Proof: By induction:

$\pi(b) = \text{root} \Rightarrow L_{min}(R_0(b)) = \frac{1}{2}$ size of cube $R_0(s)$

$\geq \frac{1}{2} L_{max} R(s)$ ✓

$\Pi(b) \neq$ root

Case V (no new min dimension)

$$L_{min}(R_0(b)) = L_{min} R_0(\Pi(b)) \underset{\text{Induct.}}{\geq} \frac{1}{2} L_{max}(R(\Pi^2(b))) \underset{\text{Split tree Const.}}{\geq} \frac{1}{2} L_{max}(R(\Pi(b)))$$

Case VI (new min. dimension)

$$L_{min}(R_0(b)) < L_{min}(R_0(\Pi(b)))$$

$$\Rightarrow L_{min}(R_0(b)) = L_i(R_0(b)) \geq L_i(R(b))$$

(min dimension = split dimension i)

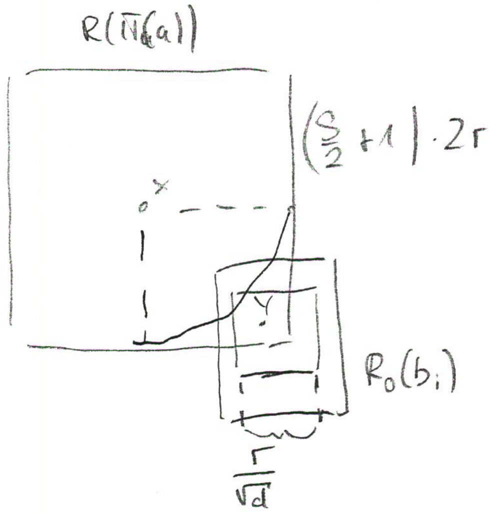
$$= \frac{1}{2} L_i(R(\Pi(b))) \underset{\text{Split } i = \text{Split dim max}}{=} \frac{1}{2} L_{max}(R(\Pi(b)))$$

$\square$

Now proof continued

$(a, b_1), (a, b_2), \ldots, (a, b_k)$

$R_0(b_i) \cap R_0(b_j) = \phi$



$$L_{max}(R(\Pi(a)) \leq L_{max}(R(\Pi(b_i)) \leq 2 \cdot L_{min}(R_0(b_i))$$

$$L_{max}(R(\Pi(a))) \leq L_{max}(R(\Pi(b_i))) \leq 2 L_m(R_0(b_i))$$

$$L_{min}(R_0(b_i)) \geq \frac{1}{2} L_{max} R(\Pi(a))$$

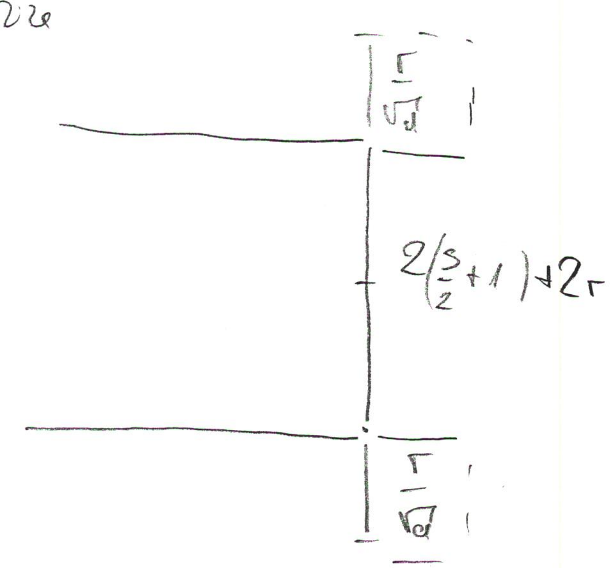
$$r = \frac{\sqrt{d}}{2} L_{max}\left(R(\bar{N}(a))\right)$$

$$\frac{r}{\sqrt{d}} = \frac{1}{2} L_{max}\, R(\bar{N}(a)))$$

$\Rightarrow$ Any $R_0(b_i)$ contains hypercube of size $\frac{r}{\sqrt{d}}$ that intersects U (move hypercube in $R_0(b_i)$ until it intersects U)

Volume of hypercube $\geq \left(\frac{r}{\sqrt{d}}\right)^d$

Union of U and all these hypercubes has size



$$2\left(\frac{s}{2}+1\right)+2r+\frac{2r}{\sqrt{d}}$$

$$\leq \left(2s+4+\frac{2}{\sqrt{d}}\right) r$$

$$\leq (2s+6)\, r$$

Volume $\leq (2s+6)^d \cdot r^d$

Hypercubes of $R_0(b_1)$ are disjunct

Volume $\geq \left(\frac{r}{\sqrt{d}}\right)^{d}$

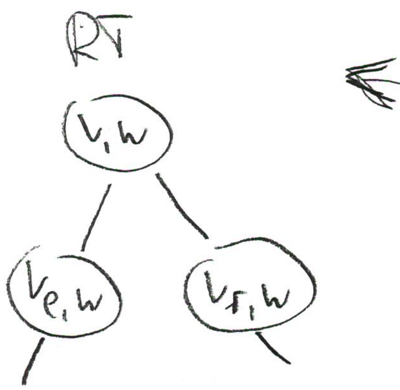
$\Rightarrow \ell = \#$ Hypercubes $\leq \frac{(2s+6)^{d}}{(\sqrt{d})^{d}} \in O(s^{d} \cdot d^{\frac{d}{2}})$

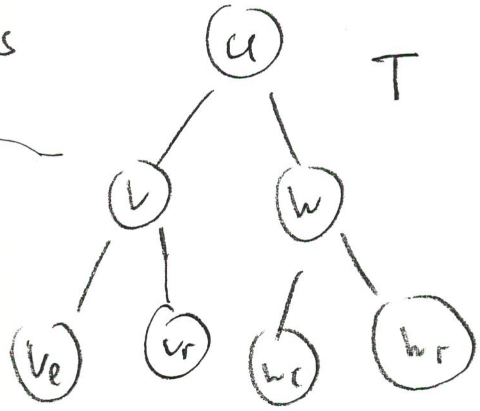
$\Rightarrow$ Analysis ② of Theorem 14

$m \in O(n \cdot s^{d} \cdot d^{\frac{d}{2}})$ ✓ $\left(O(n) \text{ nodes times } s^{d} \cdot d^{\frac{d}{2}}\right)$

Running Time of Theorem 14 $\quad$ $O(d \cdot n \log n)$ Split Tree

- Internal node $u \in T$ gives rise to "Recursion tree"
- Start of FindPairs - Compute L_{max} $O(d)$
  - Test of S_v S_w well sep. $O(d)$
- Pairs of WSPD $\rightarrow$ Leafs of RT's $\in O(s^{d} d^{\frac{d}{2}} \cdot n)$





- All FindPairs calls $\rightarrow$ all nodes of all RT's $\in O(\#\text{ of leafs of RT's}) \in O(s^{d} d^{\frac{d}{2}} \cdot n)$ $\square$