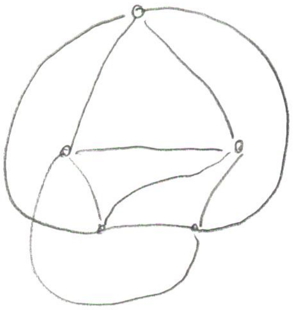
K_5 complete graph of five vertices



$cr(K_5) = 1$

Theorem 15 (Crossing number theorem)

Let $G = (V, E)$ be a simple graph (no multiple edges)

Then $cr(G) \geq \frac{1}{64} \frac{|E|^3}{|V|^2} - |V|$.

Proof: First Claim I.

planar, simple (no multiedges)

maximal planar $3|V| - 6 = |E|$, $|V| > 2$

Means: Triangulation any face has three edges (otherwise additional edge)

Induction: $|V| = 3$

✓ Show $3|V| - 6 = |E|$ for triangulation!

Assume holds for $|V| = n$

Add some vertex v $|V \cup \{v\}| = n+1$

Inside triangle: three additional edges

$3(n+1) - 6 = |E| + 3$

Claim I $\Rightarrow$ A simple planar graph has at most $3|V|-6$ edges

$\left(3|V|-6 \geq |E| \right.$ otherwise not planar $\left. \right)$

Claim II.

Crossing number simple planar: at least $|E| - 3|V|$

$|E| > 3|V|$ otherwise ✓ (at least 0)

Fewer than $|E| - 3|V|$ crossings but > 0 because of claim I

delete an edge from each crossing

delete less than $|E| - 3|V|$ edges

gives $|E|' > 3|V'|$ and no crossing (planar)

$\Downarrow$ ↯ claim I

$\square$

Proof of Theorem 15:

Drawing of $G = (V,E)$ $|V| = n$ $|E| = m$ $cr(G) = x$

$m \geq 4n$ (otherwise bound is negative, which means ok)

(Therefore $-|V|$?)

Use randomization for the proof.

Choose $p \in (0,1)$

Choose random subset V' of V: each vertex independently with probability p

$\Rightarrow$ Graph $G' = (V', E')$ for concrete instance (crossing number in the drawing of G')

$|V'| = n'$ $|E'| = m'$ x' number of crossings in G'

$E(n') = np$ $E(m') = mp^2$ and $E(x') = xp^4$

expected values for n', m', x'

Claim IV Always: $x' \geq m' - 3n'$

Therefore: $E(x') \geq E(m') - E(n')$

$\Rightarrow xp^4 \geq mp^2 - 3np$ Set $p = \frac{4n}{m} < 1$

$\Rightarrow x \geq \frac{1}{64} \frac{m^3}{n^2}$ (nice proof) $\square$

Proof of Szemerédi Trotter: $I(m,n) = O(n^{\frac{2}{3}} m^{\frac{2}{3}} + m)$

Consider set P of m points and set L of n lines that realizes the maximum number $I(m,n)$

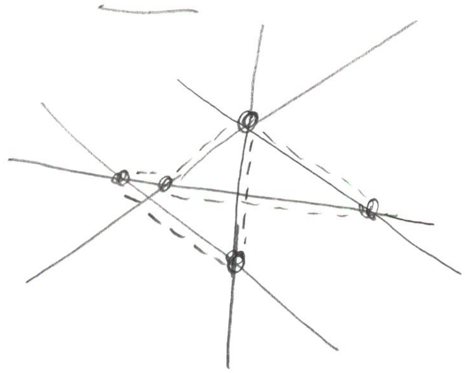
Define topological graph G. (graph with its realization)

$p \in P$ vertex in G $p, q \in P \rightsquigarrow$ edge in G if

$p, q \in \ell$ for some $\ell \in L$ next to one another

Edges are straight lines

Examples



$|E| = 6$

$|V| = 5$

$I(P, L) = 11 = |E| + |V|$

If $\ell \in L$ has $k \geq 1$ points, then G has $k-1$ edges from this line.

$\Rightarrow I(m,n) = |E| + n$ ← each line has at least one point

(k for any line with k points)

Edges are parts of lines $\Rightarrow$ $cr(G) \leq \binom{n}{2}$ (number of edges in total)

Theorem 15 $\Rightarrow cr(G) \geq \frac{1}{64} \frac{|E|^3}{m^2} - m$

$$\Leftrightarrow \frac{1}{64}\frac{|E|^3}{m^2} - m \leq cr(G) \leq \binom{n}{2}$$

$$\Rightarrow |E|^3 \leq C\left(n^2 m^2 + m^3\right)$$

$$\Rightarrow |E| \in O\left(n^{\frac{2}{3}} m^{\frac{2}{3}} + m\right)$$

$$\Rightarrow I(m,n) \in O\left(n^{\frac{2}{3}} m^{\frac{2}{3}} + m\right)$$

□

Well separated pair decomposition

- closest pair $\mathbb{R}^2$
- k closest pairs
- all nearest neighbors $\mathbb{R}^2$

} Voronoi Diagram $\Theta(n \log n)$

$\mathbb{R}^3 \rightsquigarrow \Omega(n^2)$

other concepts

Spanner

- approximation of MST
- approx. comp. dilation of N
- reducing the dilation of N
- ! construct network with low dilation → Spanner !

Point set in $\mathbb{R}^d$

Narasimhan, Smid: Geometric Spanner Networks

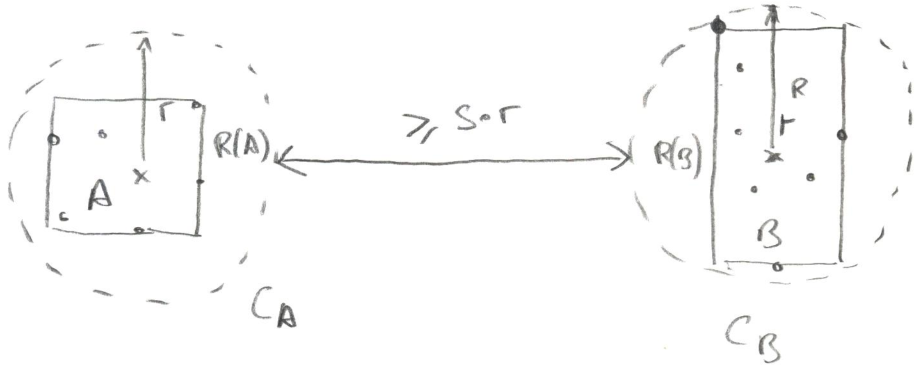
Definition 16

A pair of point sets A, B is well-separated $(A, B \in \mathbb{R}^d)$ w.r.t. s $:\Longleftrightarrow$ there are disks C_A, C_B of some radius r so that

1. $C_A \cap C_B = \phi$
2. C_A contain bounding box $R(A)$ of A
   C_B contains bounding box $R(B)$ of B
3. $|C_A C_B| \geq s \cdot r$

← min distance between points of C_A, C_B

Situation:



- Bounding Box: smallest axis parallel box that contains all points
- Usually s is much larger than r $\quad r \ll s$

Structural properties! $\quad$ (s separation ratio)

Lemma 17 $\quad$ Let $a, a' \in A$ and $b, b' \in B$

i) $|aa'| < 2r \leq \frac{2|C_A C_B|}{s} \leq \frac{2}{s}|a,b|$

(points on the same side are close, compared to points on the opposite side)

ii) $|a'b'| \leq |a'a| + |ab| + |bb'| \leq \left(1 + \frac{4}{s}\right)|ab|$

(all distances between points on opposite sides are almost equal)

Proof:

i) $a, a' \in C_A$, rad. 3. $\frac{|C_A C_B|}{s} \geq r$,

$|C_A C_B| \leq |ab|$

ii) triangle inequality, application of (i)

Idea: Represent a point set $S \in \mathbb{R}^d$ as a finite union of well-separated pairs

Definition 18 A well separated pair decomposition of S of size m for a given parameter s is a sequence of pairs $(A_1, B_1), \ldots, (A_m, B_m)$ with $A_i, B_i \subseteq S$

so that

- A_i, B_i are well separated w.r.t s $1 \leq i \leq m$
- for all $p \neq q$ in S there is a unique i so that $p \in A_i$ and $q \in B_i$ or $q \in A_i$ and $p \in B_i$

Do such things exists?

Example: Use pair $(\{a\}, \{b\}) \rightsquigarrow m = \Theta(n^2)$

Radius: $\frac{|ab|}{s+2} = r$

Application closest pair

Check all pairs $\rightsquigarrow O(n^2)$

Reduce the size of m to linear!