Case IV $a \in B$, let $B' := B \setminus \{a\} \subseteq A$

$A$ sh. by $\mathcal{F}_2 \Rightarrow$ ex. $F \in \mathcal{F}_2$ with $F \cap A = B'$

$\Rightarrow F, F \dot\cup \{a\} \in \mathcal{F}$

(Def. $\mathcal{F}_2$) and

$$B = B' \dot\cup \{a\} = (F \dot\cup \{a\}) \cap (A \dot\cup \{a\})$$

$\Rightarrow$ $A \dot\cup \{a\}$ shattered by $\mathcal{F}$ and claim holds

$$\Rightarrow \quad |\mathcal{F}_2| \leq \binom{m-1}{0} + \dots + \binom{m-1}{d-1}$$

by ind. on $d$

<u>Altogether:</u>

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2|$$
$$\leq \binom{m-1}{0} + \binom{m-1}{1} + \dots + \binom{m-1}{d} +$$
$$+ \binom{m-1}{0} + \dots + \binom{m-1}{d-1}$$
$$= 1 + \binom{m}{1} + \dots + \binom{m}{d}$$
$$= \binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{d}$$

from

$$\binom{m-1}{i} + \binom{m-1}{i-1} = \frac{(m-1)!}{i!\,(m-1-i)!} + \frac{(m-1)!}{(i-1)!\,(m-i)!}$$
$$= \frac{(m-1)!\,(m-i) + (m-1)!\,i}{i!\,(m-i)!} = \frac{m!}{i!\,(m-i)!} = \binom{m}{i}$$

$\Rightarrow$ Pard(i) ✓

□

(ii) $\dim_{VC}(\mathcal{F}) \leq d$

Show: $\forall Y \subseteq X \; |Y| = m$

$\mathcal{F}|_Y$ contains at most $\binom{m}{0} + \dots + \binom{m}{d}$ sets

apply (i) on $Y$ and $\mathcal{F}|_Y$ use $\dim_{VC}(\mathcal{F}|_Y) \leq \dim(\mathcal{F}) \leq d$ ☐

(iii) $\binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{d}$ has no closed form, due to knuth

Let $\frac{1}{2} \geq a > 0$ (fix a later)

$$1 = (a + (1-a))^m = \sum_{i=0}^{m} \binom{m}{i} a^i (1-a)^{m-i}$$

$$\geq (1-a)^m \sum_{i=1}^{d} \left(\frac{a}{1-a}\right)^i \binom{m}{i}$$

$$\geq (1-a)^m \sum_{i=1}^{d} \binom{m}{i} \left(\frac{a}{1-a}\right)^d \qquad \left(\frac{a}{1-a}\right)^i \geq \left(\frac{a}{1-a}\right)^d$$

$$\Rightarrow \sum_{i=0}^{d} \binom{m}{i} \leq \frac{(1-a)^d}{(1-a)^m a^d} =: S(a) \qquad \left[ a \leq \frac{1}{2} \Leftrightarrow \frac{a}{1-a} \leq 1 \right]$$

$S(a)$ has minimum at $a = \frac{d}{m} \left(\leq \frac{1}{2}\right)$

$$S'(a) = \frac{(1-a)^d}{(1-a)^m a^d} \Big( \ln(1-a) - \ln(a) \Big)$$

$< 0$ $\quad a < \frac{1}{2}$

mon. decreas.

$\frac{d}{m} \leq \frac{1}{2}$

(111)

$$\sum_{i=0}^{d} \binom{m}{i} \leq \underbrace{\left(1-\frac{d}{m}\right)^{d-m}}_{\dots} \left(\frac{m}{d}\right)^{d}$$

$$\ln(\dots) = (d-m)\ln\left(\frac{m-d}{m}\right)$$
$$= (d-m)\left(\ln(m-d) - \ln m\right)$$

$m \to \infty$

monotonically increasing

$\ln(\cdot)' > 0$ for $d \leq m$

$$\lim_{m\to\infty} \left(1-\frac{d}{m}\right)^{-m} = e^{d} \qquad \lim_{m\to\infty} \left(1-\frac{d}{m}\right)^{d} = 1$$

$\ulcorner$ $\ln(m) - \ln(m-d)$

$+ (d-m)\left(\frac{1}{m-d} - \frac{1}{m}\right)$ $\lrcorner$

$$\Rightarrow \frac{d}{m} \leq \frac{1}{2} \qquad \sum_{i=0}^{d} \binom{m}{i} \leq e^{d}\left(\frac{m}{d}\right)^{d}$$

Now $d \geq \frac{m}{2}$

$$f(d) = \left(\frac{em}{d}\right)^{d} \qquad \ln(f(d)) = d\left(1 + \ln m - \ln d\right)$$

monotonically increasing in $d$

$\ln(f(d))' > 0$ for $d \leq m$

$\ulcorner \left(\ln m - \ln d\right) > 0 \lrcorner$

$\Rightarrow d \in \left[\lceil \frac{m}{2} \rceil, d\right]$:

$$\sum_{i=0}^{d} \binom{m}{i} \leq \sum_{i=0}^{m} \binom{m}{i} = 2^{m} \underset{\substack{\uparrow \\ 2<e}}{<} 2^{\frac{m}{2}} \cdot e^{\frac{m}{2}} = (2e)^{\frac{m}{2}} = f\left(\frac{m}{2}\right) \leq f(d)$$

$\square$

Proof of Theorem 44 ($\varepsilon$-net Theorem) $\dim_{VC}(\mathcal{F}) = d < \infty$, $r > 2$

Existence of $\varepsilon$-net of size $C \cdot d\, r \ln r$

Proof works as follows:

Choose set $N$ of size $s := C \cdot d \cdot r \ln r$ (also fix $C$)

$\mu$-randomly from $X$ $s$ independent random draws

$\Rightarrow$ $\mu$-probability that $N$ is a $\frac{1}{r}$-net is greater than $0$. (So there has to be one! Magic)

Proof $\rightsquigarrow$ Existence, but also a bound for $C$

Preliminaries:

- Proof gives randomized algorithm for finding $\frac{1}{r}$-net
- $\mu$-random sampling
  Probability that next chosen point is $\in Y$ $= \mu(Y)$
- $s$ independent random draws, "lay back"
  Serie of size $s$ not a set!

Technical lemma distribution

Lemma 46 Let $X = X_1 + X_2 \cdots + X_n$ where $X_i$ are independent random variables $\begin{matrix} \text{Prob}(X_i = 1) = p \\ \text{Prob}(X_i = 0) = 1-p \end{matrix}$ $\forall i$

Then $\text{Prob}\left(X \geq \frac{1}{2} np\right) \geq \frac{1}{2}$ for $np \geq 8$

Deviation of value for exp. value / Value of random variable in a region around the expected value.

Proof: By Tshebyscheff-Inequality

$$P(|X - E(X)| \geq \alpha) \leq \frac{V(X)}{\alpha^2}$$

$$V(X) = E\left((X - E(X))^2\right)$$

$$E(X) = n \cdot p$$

$$V(X) \overset{\text{indep.}}{=} \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} E\left((X_i - E(X_i))^2\right) \leq np$$

$$E\left((X_i - p)^2\right) = E\left(X_i^2 - 2pX_i + p^2\right)$$

$$= E(X_i^2) - 2p\,E(X_i) + p^2$$

$$= p - p^2 \leq p$$

$$E(X_i^2) = (1-p)0^2 + p \cdot 1^2 = p$$

$$E(X_i) = (1-p)0 + p \cdot 1 = p$$

complement

$$P\left(X < \frac{np}{2}\right) \leq P\left(|X - np| \geq \frac{1}{2}np\right)$$

$$= P\left(|X - E(X)| \geq \frac{np}{2}\right) \leq \frac{np}{\left(\frac{np}{2}\right)^2} = \frac{4}{np}$$

$$P\left(X \geq \frac{np}{2}\right) \geq 1 - \frac{4}{np} \geq \frac{1}{2}$$ for $np \geq 8$. $\square$

Now proof continued:

w.l.o.g. $\mu(F) \geq \frac{1}{r}$ $\forall F \in \mathcal{F}$ (otherwise leave them out)

Two independent random draws $N, M$ of size $s$ ("lay back", Series)

Events:

$E_0$: $N$ is not an $\frac{1}{r}$-net, i.e. $\exists S \in \mathcal{F}: S \cap N = \emptyset$
(one set is missed)

$E_1$: $\exists S \in \mathcal{F}: S \cap N = \emptyset$ and $|S \cap M| \geq k$ $\quad k := \frac{s}{2r}$

( Bound $\mathrm{Prob}(E_0)$ from above by $\mathrm{Prob}(E_1)$, $\mathrm{Prob}(E_0) \overset{!}{<} 1$ )

Claim I. $\mathrm{Prob}(E_1) \leq \mathrm{Prob}(E_0)$ clear $E_1$ requires more details!
($E_0 \Rightarrow E_1$)

Claim II. $\mathrm{Prob}(E_0) \leq 2\,\mathrm{Prob}(E_1)$

Proof (Claim II): Conditional Probability

$\mathrm{Prob}(E_1 | N)$ for fixed $N$, $M$ is random

if $N$ is $\frac{1}{r}$-net $\Rightarrow \neg E_0 \wedge \neg E_1 \Rightarrow \mathrm{Prob}(E_1 | N) = 0$
$\mathrm{Prob}(E_0 | N) = 0$

if $N$ is not an $\frac{1}{r}$-net

$\Rightarrow \exists S_N \in \mathcal{F}$ with $N \cap S_N = \emptyset$

$\Rightarrow P(E_0 | N) = 1$

$\Rightarrow P(E_1 | N) \geq P(|S_N \cap M| \geq k) \geq \ldots$

( Some other set $S'$ might also fulfill $S' \cap N = \emptyset$ and $|S' \cap M| \geq k$ )

$\varepsilon = \frac{s}{2r} \qquad \mu(S_N) \geq \frac{1}{r}$

$\Rightarrow \varepsilon = \frac{s}{2r} \leq \frac{s\mu(S_N)}{2}$

$\ldots \geq \text{Prob}\left(|S_N \cap M| \geq \frac{s\mu(S_N)}{2}\right) \geq \frac{1}{2}$

$|S_N \cap M|$ random variable

$\sum_{i=1}^{s} X_i$ drawing of $M$

independent ("lay back" is important here)

$X_i = 1 \Leftrightarrow x \in S_N$ draw $x$

Probability: $x \in S_N$ is $\mu(S_N)$

$s$-draws $\Rightarrow$ Lemma 46 applicable

$(s\mu(S_N) \geq 8 \;,\; \frac{s}{r} \geq 8 \;)$

Alltogether: $\forall N \quad \text{Prob}(E_0 | N) \leq 2\, \text{Prob}(E_1 | N)$

for all fixed $N$ ✓

$\Rightarrow \quad \text{Prob}(E_0) \leq 2\, \text{Prob}(E_1)$

$P(A) = \sum_i P(B_i) \cdot P(A|B_i)$

$\Omega = \dot{\bigcup} B_i$

total probability from partial probability ✓

Now <u>different bound for</u> $Prob(E_1)$ (keeps $Prob(E_0)$ small)

There fore different drawing of $N$ and $M$ !

- Choose a series $A$ of $2s$ elements of $X$ ("lay back", prob. $\mu$)
- Choose (randomly) $s$ positions for $N$, the remaining ones build $M$

$N, M$ are choosen with the same probability as before !

Let $A$ be fixed. Consider fix $S \in \mathcal{F}$ ($N, M$ randomly)

Let $P_S := P((N \cap S = \phi \text{ and } |M \cap S| \geq k) \mid A)$

$= Prob(E_1 \mid A)$

Claim III. $P_S \leq r^{\frac{-Cd}{4}}$

small !

Proof:

Case 1 $|A \cap S| < k \Rightarrow |M \cap S| < k \Rightarrow P_S = 0$ ✓

Case 2 $|A \cap S| \geq k$

$\Rightarrow P_S \leq P(N \cap S = \phi \mid A)$

only one part

$= P$ (The $s$ positions for $N$ out of $A$ avoid ($\geq k$) positions of $S$-elements in $A$)

("positions"!)

# possibilities for s different positions if $\ell$ fixed are avoided

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$$\leq \frac{\binom{2s-\ell}{s}}{\binom{2s}{s}} = \frac{(2s-\ell)!\, s!\, \cancel{s!}}{(s-\ell)!\, \cancel{s!}\,(2s)!}$$

[ positive/avoid / all ]

# possibilities for s different pos for fixed A / # random samples of s out of 2s positions

$$= \frac{(s-\ell+1)(s-\ell+2)\cdot \ldots \qquad (2s-\ell)}{(s+1)\cdot(s+2)\cdot \quad \ldots \qquad (2s)}$$

$$\leq \left(\frac{2s-\ell}{2s}\right)^s = \left(1-\frac{\ell}{2s}\right)^s = \left(1-\frac{\frac{\ell}{2}}{s}\right)^s \underset{s\to\infty}{\leq} e^{-\frac{\ell}{2}}$$

$$\underset{|}{=} e^{-\frac{Cd\ln r}{4}} = r^{-\frac{Cd}{4}}$$

$$\left(\ell = \frac{s}{2r},\ s = Cdr\ln r\right)$$

$\dim_{VC}(F) \leq d$ was not used up to now!

Sh. Func. Lemma

$|A| \leq 2s \quad \Rightarrow \quad F|A$ has at most $\left(\frac{2es}{d}\right)^d$

different elements.

(118)

Let $A$ be fixed but $S \in \mathcal{F}$ variable

Event $\quad \underline{E}(S) := (S \cap N = \phi$ and $|S \cap M| \geq x)$

depends on $A \cap S$

$$P[\underline{E}_1 | A] \leq \sum_{\substack{A \cap S \\ \text{pairwise} \\ \text{different}}} \overbrace{P[\underline{E}(S)]}^{\substack{P_S \\ \shortparallel}}$$

"$\exists S \in \mathcal{F} : \underline{E}(S)$"

$$\leq |\mathcal{F}|_A| \cdot \max_S P_S$$

$$\leq \left(\frac{2es}{d}\right)^d r^{-\frac{Cd}{4}}$$

$$= \left(2eCr \ln r \cdot r^{-\frac{C}{4}}\right)^d < \frac{1}{2}$$

for $d, r \geq 2$ and sufficient large $C$.

Altogether: $\quad \text{Prob}(E_0) \leq 2\text{Prob}(E_1) \leq 2 \sum_A \underbrace{\text{Prob}(E_1|A)}_{<\frac{1}{2}} \underbrace{\text{Prob}(A)}_{\sum = A}$

$< 1$

Probability that a randomly choosen $N$ is not an $\frac{1}{r}$ net is smaller than $1$ $\Rightarrow$ Probability that a randomly choose net is an $\frac{1}{r}$ net is bigger than $0$ $\Rightarrow$ There exists such a net $\checkmark$

Conclusion:

Art gallery: $\dim_{VC}(\mathcal{F}) \leq 14$

$$\mathcal{F} = \{ vis_P(p) \mid p \in P \}$$

$$vis_P(p) \geq \frac{1}{10} vol(P) \qquad r = 10$$

$$\Rightarrow \quad d = 14 \quad r = 10$$

$$C \text{ so that } \left( 2e\, C \cdot 10 \ln 10 \cdot 10^{-\frac{C}{4}} \right)^{14} < \frac{1}{2}$$

$$C = 13.08$$

$$s = \underset{C}{13.08} \cdot \underset{d}{14} \cdot \underset{r}{10} \; \underset{\ln r}{\ln 10} = 4218 \text{ guards!}$$

(Art Gallery) Theorem 47 There is a constant $D > 0$ so that. For any area $P$ bounded by a simple closed Jordan curve and any point in $P$ sees at least $\frac{1}{r}$ of the volume of $P$ the number of guards required is smaller than $D \cdot r \ln r$!