

# Online Motion Planning MA-INF 1314

## Restricted Graphexploration

Elmar Langetepe  
University of Bonn

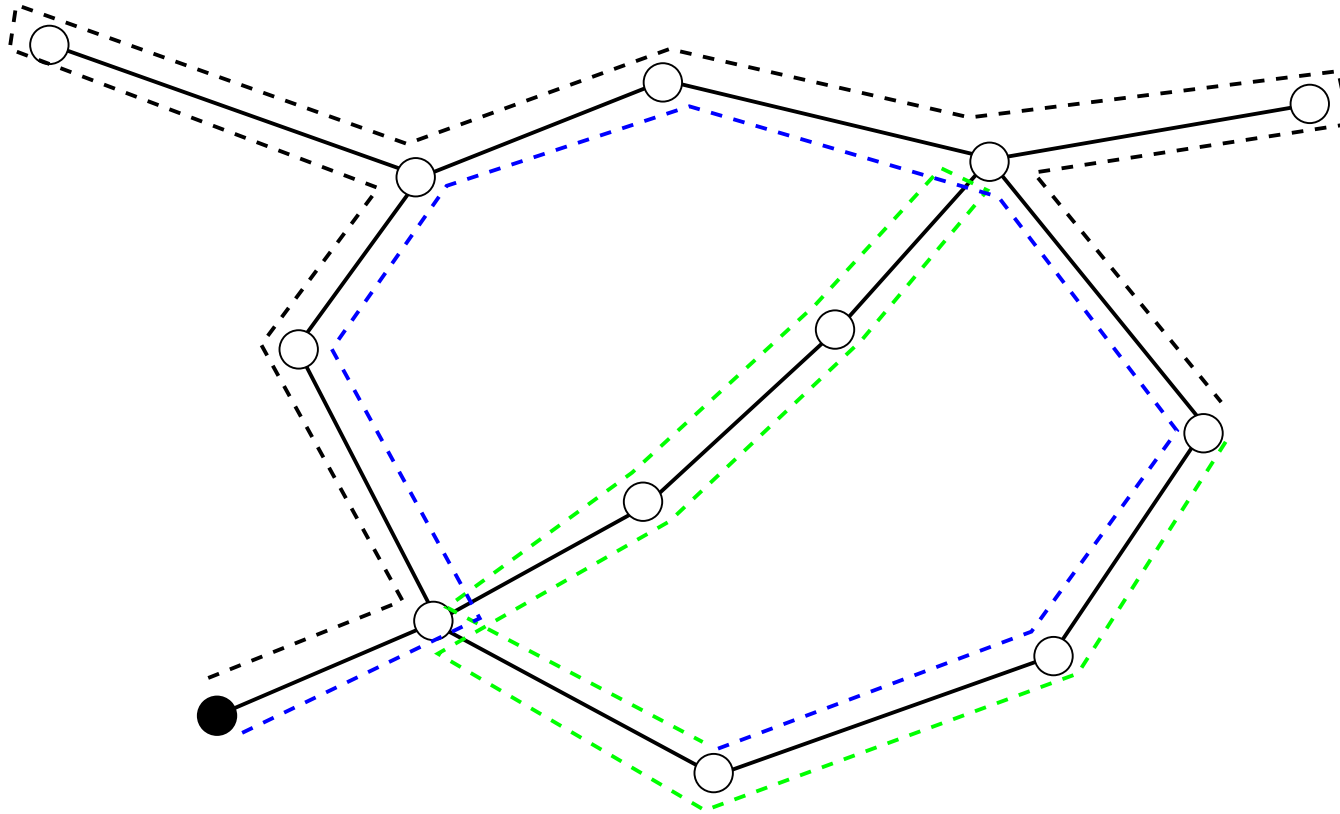
# Rep: Restricted Graphexploration

1. Agent with tether of length  $l = (1 + \alpha)r$  (i.e., cable)
2. Agent has to return after  $2(1 + \alpha)r$  steps (recharge accumulator)
3. Large graph is explored up to bounded depth  $d$ .
  - Graph has depth  $r$
  - Unit length edges
  - Small  $\alpha$

# Repetition

- Emulation: Tether variant given  $l = (1 + \alpha)r$ , cost  $T$
- **Lemma:** Accumulator-variant,  $2(1 + \beta)r$ , cost  $\frac{1+\beta}{\beta-\alpha}T$
- After  $2(\beta - \alpha)r$  break and return
- Offline Problem, Accumulator-variant, NP-hard?
- **Lemma**  $6|E|$  Offline-Approximation bei Accumulator size  $4r$
- Cut DFS in  $2r$  parts, move to, work, return and so on

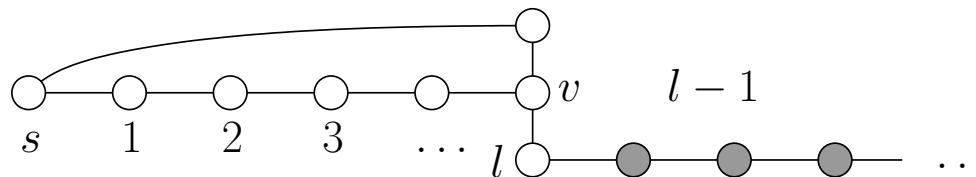
# Repetition: Offline example/Refinement



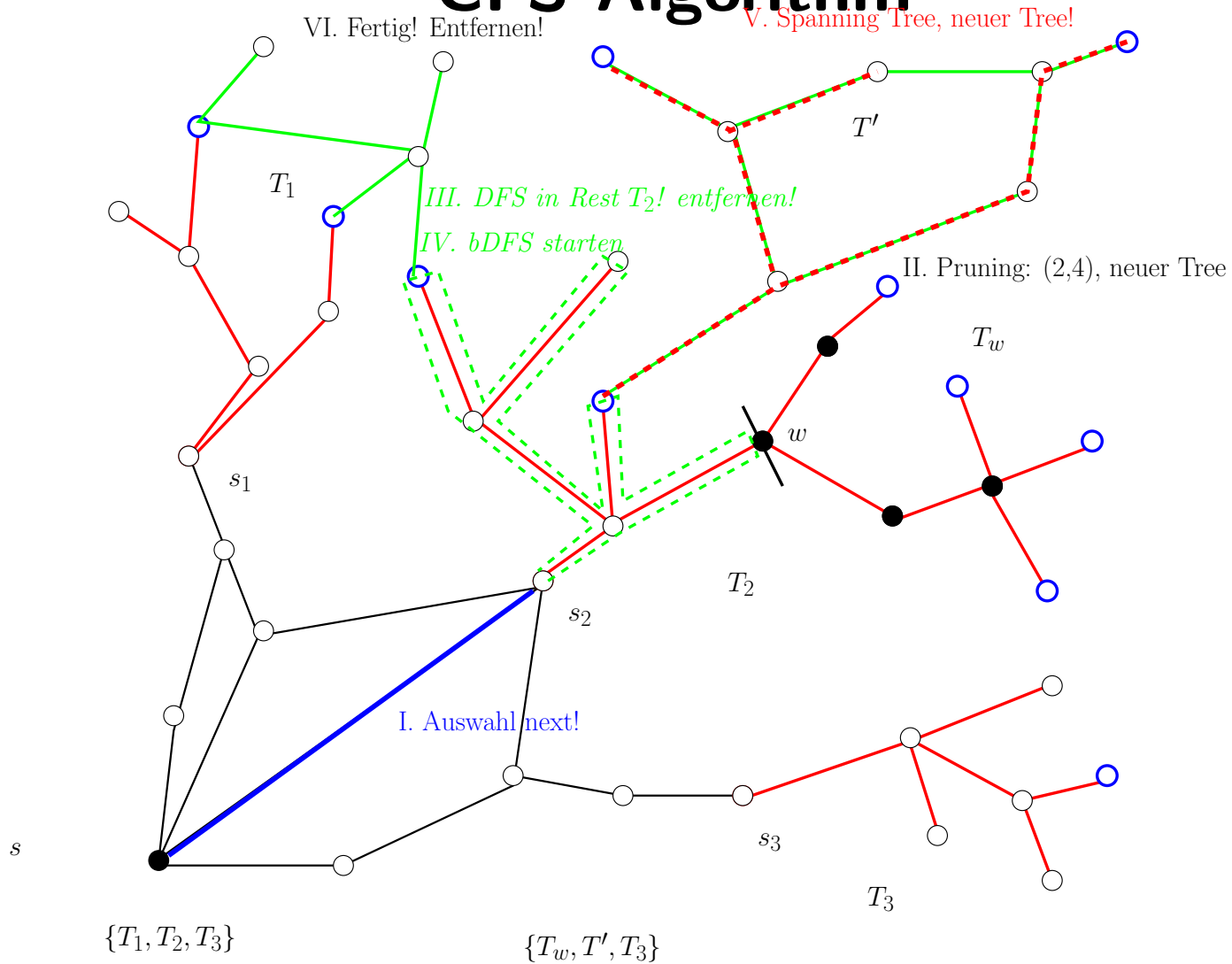
$$\left\lceil \frac{2|E|}{2r} \right\rceil \times 2r + 2|E| \leq 6|E| \quad \text{Example: } r = 5$$

# Idea: Bounded DFS

- Example unit-length edges
- $v$  fully explored, tether has ended, backtracking
- Not only bDFS



# CFS Algorithm



# CFS Algorithm

- Start bDFS at different sources ■
- Set of (edge) disjoint **trees**  $\mathcal{T} = \{ T_1, T_2, \dots, T_k \}$  ■
- Root vertices  $s_1, s_2, \dots, s_k$  ■
- Choose  $T_i$  with  $s_i$  closest to  $s$ , move to  $s_i$  ■
- Pruning of  $T_i$ : Build  $T_{w_j}$  with root  $w_j$  if: ■
  1.  $d_{T_i}(s_i, w_j) \geq \text{minDist} = \frac{\alpha r}{4}$  ■
  2.  $\text{Depth}(T_{w_j}) \geq \text{minDepth} - \text{minDist} = \frac{\alpha r}{4}$  ■
- Add all  $T_{w_j}$  to  $\mathcal{T}$ ! Remove  $T_i$  from  $\mathcal{T}$  ■
- Explore  $T_i$  without  $T_{w_j}$  from  $s_i$  by DFS and ■
- start bDFS at the incomplete vertices ■

- Graph  $G'$  of new vertices and edges ■
- Build a spanning tree  $T'$  of  $G'$  ■
- Choose root  $s'$  with minimal distance to  $s$  ■
- Add all these trees to  $\mathcal{T}$  ■
- Special case: Trees in  $\mathcal{T}$  gets fully explored ■
- Trees in  $\mathcal{T}$  with common edges are joined ■
- Merging: Build spanning tree with new root ■



# CFS Algorithm: Invariants Lemma

Execution CFS–Algorithm, properties hold: ■

- i) Any incomplete vertex belongs to a tree in  $\mathcal{T}$ . ■
- ii) There is always an incomplete vertex with  $v \in V^*$  with  $d_{G^*}(s, v) \leq r$ , until  $G^* \neq G$ . ■
- iii) For any chosen root vertex  $s_i$ :  $d_{G^*}(s, s_i) \leq r$ . ■
- iv) After pruning  $T_i$  is fully explored by DFS. All trees  $T \in \mathcal{T}$  have size  $|T| \geq \frac{\alpha r}{4}$ . ■
- v) All trees  $T \in \mathcal{T}$  are disjoint (w.r.t. edges) ■

Proof: i) and v) simply hold by construction ■

## Proof invariant ii)

$G \neq G^*$ : Ex. incomplete vertex with  $v \in V^*$  with  $d_{G^*}(s, v) \leq r$  ■

- ● Ass: For all incomplete  $v \in V^*$ :  $d_{G^*}(s, v) > r$  ■
- Choose  $v$  incomplete with minimal  $d_{G^*}(s, v)$  ■
- Shortest path to  $v$  in  $G$ :  $d_G(s, v) \leq r$  ■
- SP is partially inside  $G^*$ , first incomplete  $v'$  of  $G^*$  ■
- Inside  $V^*$  and incomplete ■
- $d_{G^*}(s, v') = d_G(s, v') \leq r < d_{G^*}(s, v)$  ■
- Contradiction to choice of  $v$ !! ■

## Proof invariant iii)

For any chosen root vertex  $s_i$ :  $d_{G^*}(s, s_i) \leq r$ .■

■ Any incomplete vertex of  $T_i$  behind  $s_i$ ■

Statement follows directly from (ii)!■

## Proof invariant iv)

After pruning  $T_i$  is fully explored by DFS.■

■ By iii) distance  $d_{G^*}(s, s_i) \leq r$ ■

Visit vertex  $v$  of  $T_i$ ■

$$r(1 + \alpha) - d_{G^*}(s, s_i) - d_{T_i}(s_i, v) \geq \frac{\alpha r}{2}$$

■

since  $d_{T_i}(s_i, v) \leq \frac{\alpha r}{2}$  by pruning■

## Proof invariant iv)

All trees  $T \in \mathcal{T}$  have size  $|T| \geq \frac{\alpha r}{4}$ . ■

- Start: bDFS from  $s$ , finished or  $|T| \geq (1 + \alpha)r > \frac{\alpha r}{4}$  ■
- Successively:  $T_i$ , so that  $s_i$  root closest to  $s$  ■
- $|T_i| \geq \frac{\alpha r}{4}$
- $T_i$  after pruning ■
- Pruning guarantees:  $|T_i| \geq \frac{\alpha r}{4}$  ■
- $T_w$  cut-off by pruning:  $|T_w| \geq \frac{\alpha r}{2} - \frac{\alpha r}{4} = \frac{\alpha r}{4}$  ■
- New bDFS-Trees:  $d_{G^*}(s, s_i) \leq r$ ,  $d_{T_i}(s_i, v) \leq \frac{\alpha r}{2}$  (pruned) ■
- At least length  $\frac{\alpha r}{2}$  of tether remains for bDFS-Trees ■
- New spanning tree  $|T'| \geq \frac{\alpha r}{4}$  ■
- All  $T \in \mathcal{T}$  have been considered ■

# Analysis Theorem

CFS–Algorithmus for restricted graph-exploration of an unknown graph with known depth  $r$  is  $(4 + \frac{8}{\alpha})$ –competitive. ■

Subtree  $T_R$  ■

- Subtree  $T_R$ , cost ■
- $K_1(T_R)$ : path from  $s$  to  $s_i$  in  $G^*$  ■
- $K_2(T_R)$ : Exploration by DFS ■
- $K_3(T_R)$ : bDFS starting from incomplete vertices (Graph!) ■
- $\sum_{T_R} K_3(T_R) \leq 2 \cdot |E|$ , bDFS only visits unexplored edges ■
- $\sum_{T_R} K_2(T_R) = \sum_{T_R} 2 \cdot |T_R| \leq 2 \cdot |E|$ , the cost of DFS ■

# Analysis Theorem

- Subtree  $T_R$ , cost
- ●  $K_1(T_R)$ : path from  $s$  to  $s_i$  in  $G^*$  ■
- $K_1(T_R) = 2 \cdot d_{G^*}(s, s_i) \leq 2r$ , **Lemma** ■
- Pruning: Komplexity  $T_R$  at least  $\frac{\alpha r}{4}$  ■
- $|T_R| \geq \frac{\alpha r}{4}$ , which gives  $r \leq \frac{4|T_R|}{\alpha}$ , **Lemma** ■
- $\sum_{T_R} K_1(T_R) \leq \sum_{T_R} 2r \leq \frac{8}{\alpha} \sum_{T_R} |T_R| \leq \frac{8}{\alpha} |E|$  ■
- $2 \cdot |E| + 2 \cdot |E| + \frac{8}{\alpha} |E|$  versus  $|E|$  ■
- $\leq (4 + \frac{8}{\alpha}) |E|$  ■

## Corollary

CFS–Algorithm for restricted graph-exploration of an unknown graph of known depth requires  $\Theta(|E| + |V|/\alpha)$  steps. ■

- $K_3(T_R)$ : bDFS starting from incomplete vertices, visit edges ■
- $K_1(T_R)$  and  $K_2(T_R)$  analysed by TREES  $T_R$  ■
- vertices and edges same size, vertex could appear in 2 trees! ■
- $(2 + \frac{8}{\alpha}) 2|V|$  and  $2|E|$  only for bDFS ■



# Unknown graph of unknown depth!

- Assumption: Depth  $R$  is not known■
- Heuristik: Double  $r$  successively ■
- So that finally  $r$  is large enough!■
- Start with  $r := 2$ , ■  $\approx \log_2 R$  calls■
- Means  $O(\log R |E|)$  steps■
- Re-exploration by bDFS could be avoided, trees!■
- $O(|E| + (\log R)|V|)$  steps, **Corollary**■

# Unknown graph of unknown depth $R!$

- Improvement! ■
- Adjusting pruning and explore steps ■
- Depth and tether by circumstances:  $d_{G^*}(s, s_i)$  is current  $r$  ■
- Algorithm still has cost:  $(2 + \frac{8}{\alpha}) 2|V| + 2|E|$  ■
- **prune** $(T_i, s_i, \frac{\alpha r}{4}, \frac{\alpha r}{2})$  ■
- Substitution by: **prune** $(T_i, s_i, \frac{\alpha d_{G^*}(s, s_i)}{4}, \frac{9\alpha d_{G^*}(s, s_i)}{16})$  ■
- **explore** $(\mathcal{T}, T_i, s_i, (1 + \alpha)r)$  ■
- Substitution by: **explore** $(\mathcal{T}, T_i, s_i, (1 + \alpha)d_{G^*}(s, s_i))$  ■

## $d_{G^*}(s, s_i)$ substitute of $r$ for explore/prune

- Start-problem:  $d_{G^*}(s, s_i) = 0$ ,  $r := \max(d_{G^*}(s, s_i), c)$  ■
- Structural properties **Lemma** ■
  - i) Any incomplete vertex belongs to a tree in  $\mathcal{T}$ . ■
  - ii) There is always an incomplete vertex with  $v \in V^*$  with  $d_{G^*}(s, v) \leq r$ , until  $G^* \neq G$ . ■
  - iii) For any chosen root vertex  $s_i$ :  $d_{G^*}(s, s_i) \leq r$ . ■
  - iv) After pruning  $T_i$  is fully explored by DFS. All trees  $T \in \mathcal{T}$  have size  $|T| \geq \frac{\alpha r}{4}$ . ■
  - v) All trees  $T \in \mathcal{T}$  are disjoint (w.r.t. edges) ■
- 
- i), ii), iii) and v) also true for adjusted calls ■

# Properties for adjusted explore/prune

$r := \max(d_{G^*}(s, s_i), c)$  where  $d_{G^*}(s, s_i)$  minimal for  $T_i \in \mathcal{T}$  ■

## Lemma:

- i) Any incomplete vertex belongs to a tree in  $\mathcal{T}$ . ■
  - ii) There is always an incomplete vertex with  $v \in V^*$  with  $d_{G^*}(s, v) \leq r$ , until  $G^* \neq G$ . ■
  - iii) For any chosen root vertex  $s_i$ :  $d_{G^*}(s, s_i) \leq r$ . ■
  - iv) After pruning  $T_i$  is fully explored by DFS. All trees  $T \in \mathcal{T}$  have size  $|T| \geq \frac{\max(d_{G^*}(s, T), c)\alpha}{4}$ . ■
  - v) All trees  $T \in \mathcal{T}$  are disjoint (w.r.t. edges) ■
- iv) has to be shown! ■

## Proof invariant iv)

After pruning:  $T_i$  is fully explored by DFS

Distance  $d_{G^*}(s, s_i)$

Visit vertex  $v$  of  $T_i$

$$(1 + \alpha)d_{G^*}(s, s_i) - d_{G^*}(s, s_i) - d_{T_i}(s_i, v) \geq \frac{7d_{G^*}(s, s_i)\alpha}{16}$$

since  $d_{T_i}(s_i, v) \leq \frac{9d_{G^*}(s, s_i)\alpha}{16}$  by pruning!

## Proof invariant iv)

$$\forall T \in \mathcal{T} : |T| \geq \frac{\max(d_{G^*}(s, T), c)\alpha}{4}$$

- Start: bDFS from start, finished or  $|T| \geq (1 + \alpha)c > \frac{\alpha c}{4}$
- Otherwise  $T_i$ , with  $s_i$  root closest to  $s$ , Ass.:  $d_{G^*}(s, T_i) > c$
- Show:  $|T_w| \geq \frac{d_{G^*}(s, T_w)\alpha}{4}$
- Tree  $T_i$  after pruning,  $d_{G^*}(s, T_i) = d_{G^*}(s, s_i)$
- Pruning guarantees:  $|T_i| \geq \frac{d_{G^*}(s, T)\alpha}{4}$
- $T_w$  cut-off by pruning:  $|T_w| \geq \frac{9d_{G^*}(s, s_i)\alpha}{16} - \frac{d_{G^*}(s, s_i)\alpha}{4} = 5\frac{d_{G^*}(s, s_i)\alpha}{16}$
- $d_{G^*}(s, T_w) \leq d_{G^*}(s, s_i) + d_{G^*}(s_i, w) = (1 + \frac{\alpha}{4})d_{G^*}(s, s_i) < \frac{5d_{G^*}(s, s_i)}{4}$ ,  $\alpha < 1$
- Gives:  $|T_w| > \frac{d_{G^*}(s, T_w)\alpha}{4}$

## Proof invariant iv)

$$\forall T \in \mathcal{T} : |T| \geq \frac{\max(d_{G^*}(s, T), c)\alpha}{4}$$

- Spanning trees  $T'$  after bDFS from  $v$  in  $T_i$
- bDFS-trees after  $v$ :  $d_{G^*}(s_i, v) \leq \frac{9\alpha d_{G^*}(s, s_i)}{16}$
- $d_{G^*}(s, T') \leq d_{G^*}(s, s_i) + d_{G^*}(s_i, v) < \frac{25d_{G^*}(s, s_i)}{16}$
- $d_{G^*}(s, s_i) > \frac{16d_{G^*}(s, T')}{25}$
- $T'$  incomplete
- $d_{G^*}(s_i, v) \leq \frac{9\alpha d_{G^*}(s, s_i)}{16}$
- At least  $\frac{7\alpha d_{G^*}(s, s_i)}{16}$  tether rest
- $|T'| \geq \frac{7\alpha d_{G^*}(s, s_i)}{16} > \frac{7\alpha d_{G^*}(s, T')}{25} > \frac{d_{G^*}(s, T')\alpha}{4}$
- Either explored or  $|T'| > \frac{d_{G^*}(s, T')\alpha}{4}$
- Any  $T \in \mathcal{T}$  was considered

# Analysis Theorem

CFS–Algorithmus for restricted graph-exploration of unknown graph with unknown depth is  $(4 + \frac{8}{\alpha})$ –competitive for  $0 < \alpha < 1$ . ■

Trees  $T_R$  ■

- Tree  $T_R$ , cost ■
- $K_1(T_R)$ : path from  $s$  to  $s_i$  in  $G^*$  ■
- $K_2(T_R)$ : Exploration by DFS ■
- $K_3(T_R)$ : bDFS from incomplete vertex (Graph!) ■
- $\sum_{T_R} K_3(T_R) \leq 2 \cdot |E|$ , since bDFS visits only unexplored edges ■
- $\sum_{T_R} K_2(T_R) = \sum_{T_R} 2 \cdot |T_R| \leq 2 \cdot |E|$ , cost for DFS ■



# Analyse Theorem 1.30

- Teilbaum  $T_R$ , Kosten
- $K_1(T_R)$ : path from  $s$  to  $s_i$  in  $G^*$
- $K_1(T_R) = 2 \cdot d_{G^*}(s, s_i) \leq \frac{8|T_R|}{\alpha}$
- $|T_R| \geq \frac{d_{G^*}(s, T_R)\alpha}{4} = \frac{d_{G^*}(s, s_i)\alpha}{4}$ , **Lemma iv)**
- $\sum_{T_R} K_1(T_R) \leq \sum_{T_R} 2d_{G^*}(s, s_i) \leq \frac{8}{\alpha} \sum_{T_R} |T_R| \leq \frac{8}{\alpha} |E|$
- $2 \cdot |E| + 2 \cdot |E| + \frac{8}{\alpha} |E|$  against  $E$

## Corollary

CFS–Algorithmus for restricted graph-exploration of unknown graph with unknown depth performs  $\Theta(|E| + |V|/\alpha)$  steps for  $0 < \alpha < 1$ . ■