3. General Theoretical Foundation

Conflict Graph G(V, E)

- V: configurations in $H(N^i)$ and objects in $N \setminus N^i$
- E: conflict relations between configurations in $H(N^i)$ and objects in $N\setminus N^i$
- 1. Use the conflict graph to find out all configurations in ${\cal H}(N^i)$ which conflict with S^{i+1}
- 2. Create new configrations defined (or called supported) by S^{i+1}
- 3. Update conflict graph
 - Remove invalid configurations and the corresponding edges
 - Add edges between the new configuations in $H(N^{i+1})$ and their conflicted objectes in $N\setminus N^{i+1}$

Hisotry graph G(V, E) (directed graph)

- V: configurations in $H(N^0)$, $H(N^1)$, ..., $H(N^i)$
- E: direct arcs from $H(N^{j-1}) \setminus H(N^j)$ and $H(N^j) \setminus H(N^{j-1})$, for $1 \leq j \leq i$, i.e., configurations killed by S^j and configurations created by S^j
 - $-\;G$ is an acyclic graph, and only configurations in $H(N^0)$ don't have in-going edges and are called roots.
 - If an object S conflicts with a configuration f, there is one path from a root to f along which all configurations are in conflict with S.
 - (optional) Each configuration has a constant number of out-going edges.
- 1. Use the history to find out all configurations in ${\cal H}(N^i)$ in conflict with S^{i+1}
- 2. Create new configrations defined (or called supported) by S^{i+1}
- 3. Add edges between $H(N^i) \setminus H(N^{i+1})$ and $H(N^{i+1}) \setminus H(N^i)$

Kenneth L. Clarkson, Kurt Mehlhorn, and Raimund Seidel Four, Results on Randomized Incremental Construction Computational Geometry: Theory and Applications 3, pp. 185–212, 1993.

Denotation Changes

N	S
S_1, S_2, \ldots, S_n	$\pi_S = x_1, x_2, \dots, x_n$
H(N)	$\mathcal{F}_0(S)$
history(i)	H(i)

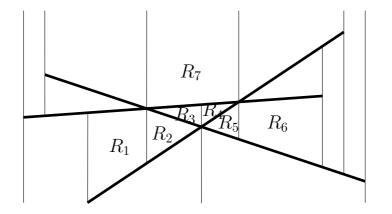
- 3.1 Basic Denotations
 - S: a set of n objects (points, line segments, circles)

 $\mathcal{F}(S)$: configurations defined by S

- A configuration is defined by at most b objects.
 - a triangle is defined by 3 points, a trapezoid is defined by at most 4 line segments.
- A multiset: $c \leq b$ elements can define more than one configuration

- 3 segments can defined 7 trapezoids

• For a configuration $F \in \mathcal{F}(S)$ and an object $x \in S$, if $x \in F$, F replies on x and x supports F



 $C \subseteq S \times \mathcal{F}(S)$: conflict relations between S and $\mathcal{F}(S)$

- $(x, F) \in C \to x$ does not support F
- $(x, F) \in C$ usually means a nonempty intersection between x and F – a point x insides a triangle F

Example: Vertical Trapezoidal decomposition

- S: a set of n line segment
- $\mathcal{F}(S)$: trapezoids defined by S (two trapezoids can intersect)
- $(x, F) \in C$: line segment x intersects F

– Different from that an endpoint of x is located inside a trapezoid F

 $\mathcal{F}_0(\mathbf{R}) = \{F \in \mathcal{F}(R) \mid \forall x \in R, (x, F) \notin C\}, \text{ for a } r\text{-element random sample } R \text{ of } S$

• any configuration in $\mathcal{F}_0(R)$ does not conflict with any object in R.

$$\pi = (x_1, x_2, \ldots, x_n)$$
 is a random permutation of S

- $R_j = \{x_1, x_2, \ldots, x_j\}$
- $\pi_r = (x_1, x_2, \dots, x_j)$

History $H_r(\pi) = H(x_1, x_2, \dots, x_r) = \bigcup_{1 \le i \le r} \mathcal{F}_0(R_i)$

- (x_1, x_2, \ldots, x_r) is the first r elements of π_S
- equivalent to trapzoids in history(r)
- $H_r = H_r(\pi)$

Fact

If $\pi = (x_1, x_2, \ldots, x_n)$ is a random permutation of S, R_j is a random subset of size j of S, (x_1, x_2, \ldots, x_j) is a random permutation of R_j , x_j is a random element of R_j , and if δ is a (fixed) permutation, $\pi\delta$ is random permutation For a subset $R \subseteq S$, r = |R|, and two distinct objects, $x, y \in R$,

• $\deg(x, R) = |\{F \in \mathcal{F}_0(R) \mid x \text{ supports } F\}|$

– the number of triangles in a triangulation incident to a point \boldsymbol{x}

• $pdeg(x, y, R) = |\{F \in \mathcal{F}_0(R) \mid x \text{ and } y \text{ support } F\}|$

– the number of triangles in a triangulation is with an edge \overline{xy}

•
$$c(R) = \frac{1}{r} \sum_{x \in R} \deg(x, R)$$

• $p(R) = \frac{1}{r(r-1)} \sum_{(x,y) \in R \times R} \operatorname{pdeg}(x, y, R)$

Important Expected Values

- $c_r = E[c(R)] = \sum_{R \subseteq S, |R| = r} c(R) / {n \choose r}$
- $p_r = E[p(R)] = \sum_{R \subseteq S, |R|=r} p(R) / {n \choose r}$
- $f_r = \sum_{R \subseteq S, |R|=r} |\mathcal{F}_0(R)| / {n \choose r}$
- $c_1 = p_1 = f_1$ and for j < 1 or j > n, $c_j = p_j = f_j = 0$.

3.2 Lemmas and Theorems

All expected values are computed with respect to a random permutation $\pi = (x_1, x_2, \dots, x_n)$ of S

Lemma 1

- 1. $c_r \leq bf_r/r$
- 2. $p_r \le b(b-1)f_r/r(r-1)$, for r > 1

proof: For every configuration $F \in \mathcal{F}_0(S)$

- 1. At most b objects support F
- 2. At most b(b-1) order pairs of objects support F

Theorem 1

Let C_r be the expected size of H_r . $C_r = \sum_{1 \le i \le r} c_i$. proof:

- 1. H_0 is empty and $C_0 = 0$
- 2. For $i \ge 1$, $|H_i \setminus H_{i-1}| = \deg(x_i, R_i)$.
- 3. R_i is a random subset of S of size i and x_i is a random element of R_i , $E[\deg(x_i, R_i)] = E[c(R_i)] = c_i.$
- 4. $E[|H_r|] = E[\sum_{1 \le i \le n} |H_i \setminus H_{i-1}|] = \sum_{1 \le i \le n} |E[|H_i \setminus H_{i-1}|] = \sum_{1 \le i \le r} c_i$

Example Let R be a random subset of S of size r

- Since the triangualtion of R has O(r) triangles, $c_r = O(1)$ and $E[|H_r|] = O(r)$.
- Since the expected number of trapezoids in the trapezoidal decomposition of R is $O(r+kr^2/n^2)$, where k is the number of intersections among the n line segments, $c_r = O(1 + kr/n^2)$ and $E[|H_r|] = O(r + kr^2/n^2)$

Theorem 2

The expected number of configurations in H_{r-1} which are in conflict with x_r is $-c_r + \sum_{j \leq r} p_j$. proof:

- Let X be the number of configurations $F \in H_{r-1}$ with $(x_r, F) \in C$
- Let $H = H_{r-1} = H(x_1, x_2, \dots, x_{r-1})$ Let $H' = H(x_r, x_1, \dots, x_{r-1})$, i.e., x_r is pretend to be inserted first.
- $\bullet \ |H\cup H'| = |H| + |H' \setminus H| = |H'| + |H \setminus H'|$
- $\bullet \ X = |H \setminus H'|$
- $H' \setminus H$ comprises configurations supported by x_r . How many of them appear when x_j is inserted, $1 \leq j \leq r-1$. Let $R'_j = R_j \cup \{x_j\}$. For each $F \in H' \setminus H$,
 - either $F \in \mathcal{F}_0(\{x_r\})$ or
 - $-F \in \mathcal{F}_0(R'_j)$ and x_j support $F, \exists j \ge 1$. Since F must be supported by x_r , the total number is $pdeg(x_r, x_j, R'_j)$

•
$$X = |H| - |H'| + |H' \setminus H|$$

= $|H| - |H'| + |\mathcal{F}_0(\{x_r\})| + \sum_{1 \le j \le r-1} \operatorname{pdeg}(x_r, x_j, R'_j)$
 $E[X] = E[|H|] - E[|H'|] + E[|\mathcal{F}_0(\{x_r\})|] + \sum_{1 \le j \le r-1} E[\operatorname{pdeg}(x_r, x_j, R'_j)]$

- $E[|H|] = C_{r-1}, E[|H'|] = C_r$, and $C_{r-1} C_r = -c_r$
- $E[|\mathcal{F}_0(\{x_r\})|] = f_1 = p_1$ and $E[pdeg(x_r, x_j, R'_j)] = p_{j+1}$ since R'_j is a random subset of S of size j + 1 and x_r and x_j are random elements of this subset
- $E[X] = -c_r + \sum_{j \le r} p_j$

Example: Vertical Trapezoidal Decomposition

• $c_i \le bf_i/i = 4 * O(i + ki^2/n^2)/i = O(1 + ki/n^2)$

•
$$p_i \leq b(b-1)f_i/i(i-1) = 12O(i+ki^2/n^2)/i(i-1) = O(1/i+k/n^2)$$

• $-O(1 + ki/n^2) + \sum_{1 \le i \le r} O(1/i + k/n^2) = O(\log r + kr^2/n^2)$

Lemma 2

- 1. The expected number of configurations in $\mathcal{F}_0(R_{j-1})$ in conflict with x_r is $f_{j-1} f_j + c_j$
- 2. The expected number of configurations in $\mathcal{F}_0(R_{j-1})$ supported by x_{j-1} and in conflict with x_r is at most $b(f_{j-1} - f_j + c_j)/(j-1)$

proof

- 1. Difference between $\mathcal{F}_0(R)$ and $\mathcal{F}_0(R \cup \{x\})$
 - configurations in $\mathcal{F}_0(R)$ in conflict with x
 - configuration in $\mathcal{F}_0(R \cup \{x\})$ supported by x

 $\begin{aligned} \mathcal{F}_0(R_{j-1} \cup \{x_r\}) &= \mathcal{F}_0(R_{j-1}) \setminus \{F \in \mathcal{F}_0(R_{j-1}) \mid (x_r, F) \in C\} \cup \{F \in \mathcal{F}_0(R_{j-1} \cup \{x_r\}) \mid x_r \text{ supports } F\} \\ &\to E[|\mathcal{F}_0(R_{j-1})|] - E[|\mathcal{F}_0(R_{j-1} \cup \{x_r\})|] + E[|\{F \in \mathcal{F}_0(R_{j-1} \cup \{x_r\}) \mid x_r \text{ supports } F\}|] = f_{j-1} - f_j + c_j \end{aligned}$

2. Since x_{j-1} is a random element of R_{j-1} , the probability with which a configuration in (1) is supported by x_{j-1} is at most b/(j-1), implying an expected value $b(f_{j-1} - f_j + c_j)/(j-1)$

Conflict History

- $G = G_n = G_\pi = C \cap (S \times H_n)$ for a random sequence π of S, i.e., the conflict relations between S and H_n .
- Bipartite Graph G(U, V, E)

$$-U = S$$

$$-V = H_n$$

$$-E = \{(u, v) \mid u \in U, v \in V, (u, v) \in C\}$$

$$|G| = |E|$$

Theorem 3

$$E[|G|] = -C_n + \sum_{1 \le j \le n} (n-j+1)p_j.$$
proof

$$E[|G|] = \sum_{1 \le i \le n} (-c_i + \sum_{1 \le j \le i} p_j)$$

$$= -C_n + \sum_{1 \le i \le n} \sum_{1 \le j \le i} p_j$$

$$= -C_n + \sum_{1 \le j \le n} (n-j+1)p_j \text{ since } p_j \text{ occurs } (n-j+1) \text{ times}$$

Example Vertical Trapezoidal Decomposition

- $C_n = \sum_{1 \leq i \leq n} O(i + ki/n^2) = O(n+k)$
- $\bullet \ |G| \leq \sum_{1 \leq i \leq n} (n-i+1)O(1/i+k/n^2) \\ \leq \sum_{1 \leq i \leq n} O(n/i+k/n) = O(n\log n+k)$
- note that a conflict relation between a segment x and a trapezoid F indictes that x intersect F (not defined for an endpoint of x)

3.3 Deletion

For
$$\pi = (x_1, x_2, \dots, x_n) \in \Pi_S$$
 and $i \in [1 \cdots n]$,
 $\pi \setminus i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

Delete x_i from π as x_i has never been inserted.

- Compute $H(\pi \setminus i)$ from $H(\pi)$
- Analyze $G(\pi \setminus i)$ from $G(\pi)$

Theorem 4 $\frac{1}{n!n} \sum_{\pi \in \Pi_S} \sum_{1 \le i \le n} |H(\pi) \oplus H(\pi \setminus i)| \le 2b \frac{C_n}{n} - c_n.$ proof

- $|B \oplus A| = |A| |B| + 2|B \setminus A|$ $|H \oplus H(\pi \setminus i)| = |H(\pi \setminus i)| - |H| + 2|H \setminus H(\pi \setminus i)|$
- $H \setminus H(\pi \setminus i)$ comprises configurations in H supported by x_i
 - $-E[|H|] = C_n$, and any $F \in H$ is supported by no more than b objects

$$-E[|H \setminus H(\pi \setminus i)|] \le bC_n/n$$

• $E[|H(\pi) \oplus H(\pi \setminus i)|] = C_{n-1} - C_n + 2E[|H \setminus H(\pi \setminus i)|] \le -c_n + 2bC_n/n$

Example: Vertical Trapezoidal Decomposition

- $C_n = O(k+n), b = 4, \text{ and } c_i = O(1+ki^2/n^2)$
- $\bullet \ E[|H \oplus H(\pi \setminus i)|] = O(1+k/n)$

Theroem 5

$$\begin{split} E[|G(\pi \setminus i) \setminus G(\pi)|] &= \frac{1}{n!n} \sum_{\pi \in \Pi_S} \sum_{1 \le i \le n} |G(\pi \setminus i) \setminus G(\pi)| \\ &\le c_n - (b+1)C_n/n + \sum_{1 \le j \le n} bp_j - \sum_{1 \le j \le n} (b+1)(j-1)p_j/n. \\ proof \end{split}$$

- $G = G(\pi), |G(\pi \setminus i) \setminus G| = |G(\pi \setminus i)| |G| + |G \setminus G(\pi \setminus i)|$ $\rightarrow E[|G(\pi \setminus i) \setminus G|] = E[|G(\pi \setminus i)|] - E[|G|] + E[|G \setminus G(\pi \setminus i)|]$ $\rightarrow E[|G(\pi \setminus i) \setminus G|] = E[|G \setminus G(\pi \setminus i)|] + c_n - \sum_{1 \le j \le n} p_j$
- A pair (x, F) is in $G \setminus G(\pi \setminus i)$ if it is in G and either $x_i = x$ or $x_i \in F$. \rightarrow at most b + 1 choices of x_i \rightarrow the probablity with $(x, F) \in G \setminus G(\pi \setminus i)$ is b + 1/n
- $\bullet \ E[|G \setminus G(\pi \setminus i)|] \leq (b+1)E[|G|]/n$

Example: Vertical Trapezoidal Decomposition

• $E[|G \setminus G(\pi \setminus i)|] = O(\log n + k/n)$

Theroem 6

For a fixed *i*, let *I* be the set of conflicts of the form (x_j, F) with j > i and $F \in \mathcal{F}_0(R_{i-1}) \setminus \mathcal{F}_0(R_i)$. Then for random $\pi \in \Pi_S$ and random $i \in [1 \cdots n]$, $E[|I|] = (E[|G|] - E[|H|] + f_n)/n$ proof

- Let I_i denote the set I for $x_i \to E[|I|] = \sum_{1 \le i \le n} E[|I_i|]/n$
- Since I_i are disjoint, $E[I] = E[|\bigcup_i I_i|]/n$
- For any conflict $(x_j, F) \in G$,
 - either $F \in \mathcal{F}_0(R_{j-1})$

- or there is exactly one i < j such that $F \in \mathcal{F}_0(R_{i-1}) \setminus \mathcal{F}_0(R_i)$ $\rightarrow (x_j, F) \in I_i$

- $E[|G|] = E[|\bigcup_{1 \le i \le n} I_i|] + E[|\{(x_j, F) \in G \mid F \in \mathcal{F}_0(R_{j-1})\}|]$
- For each conflict (x_j, F) with $F \in \mathcal{F}_0(R_{j-1}), F$ appears in $H \setminus \mathcal{F}_0(S)$ exactly once $\to E[|\{(x_j, F) \in G \mid F \in \mathcal{F}_0(R_{j-1})\}|] = E[|H|] - |\mathcal{F}_0(S)|$

Example: Vertical Trapezoidal Decomposition

• $E[|I|] = O(\log n + k/n)$