Online Motion Planning MA-INF 1314
Smart DFS

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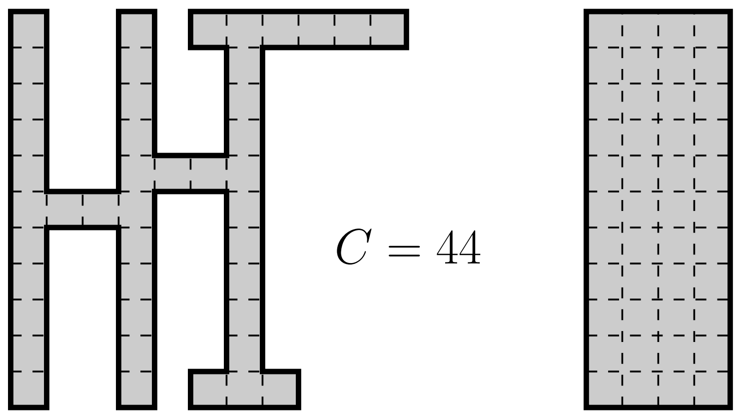
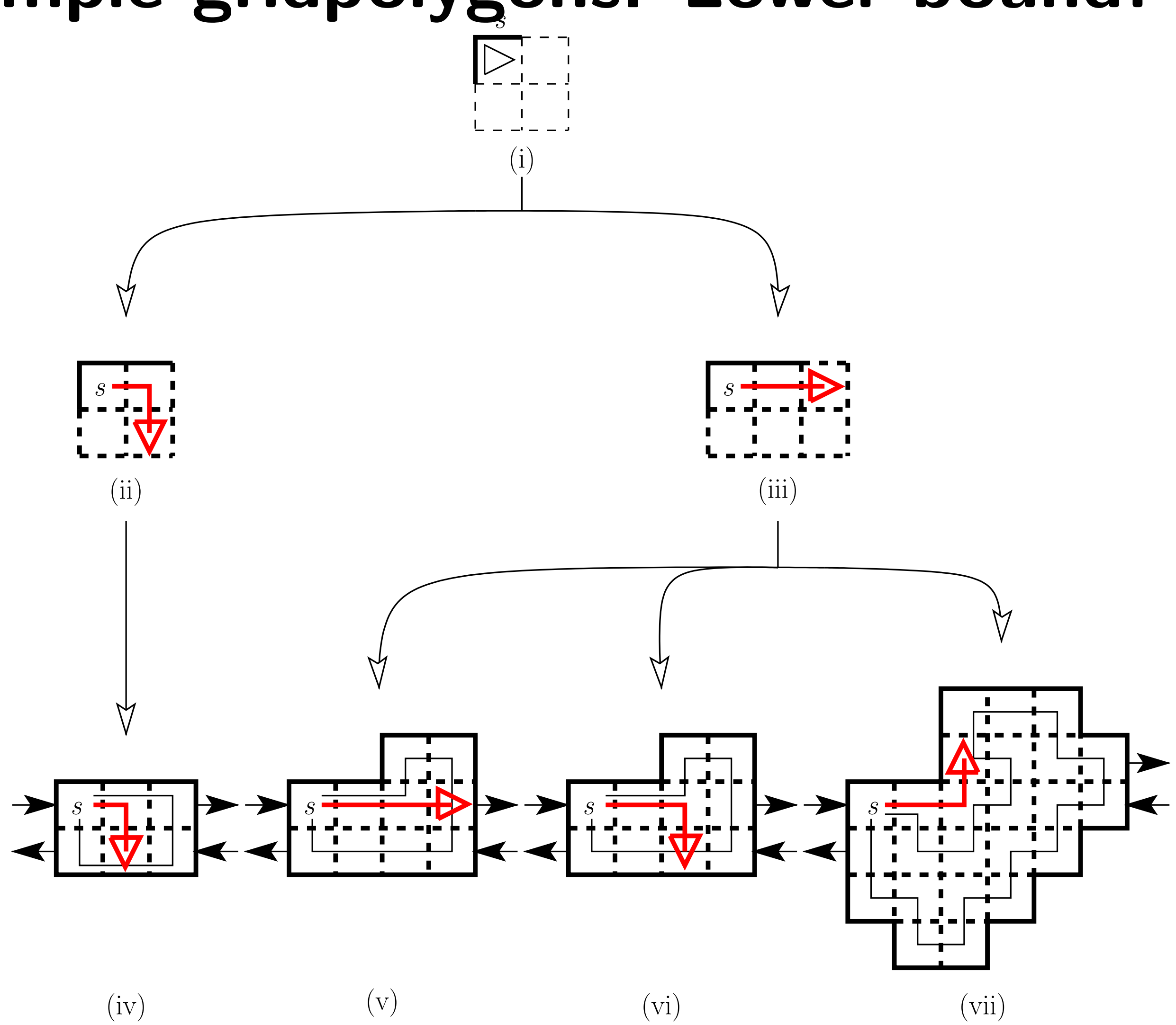
Repetition: Simple gridpolygons

- Pure DFS is not the best idea
- Realation between edges and cells



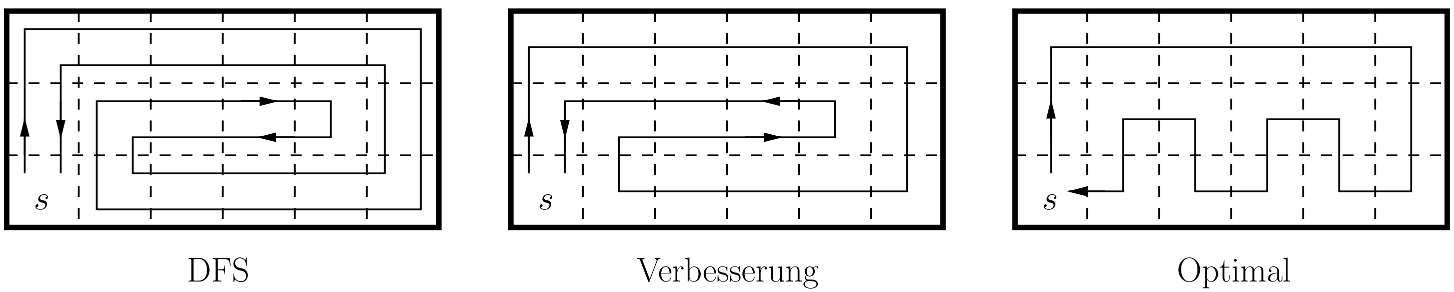
$E = 90 = 2C + 2$ $\qquad$ $E = 30 \ll 2C$

Simple gridpolygons: Lower bound! $\frac{7}{6}$!



Improve DFS

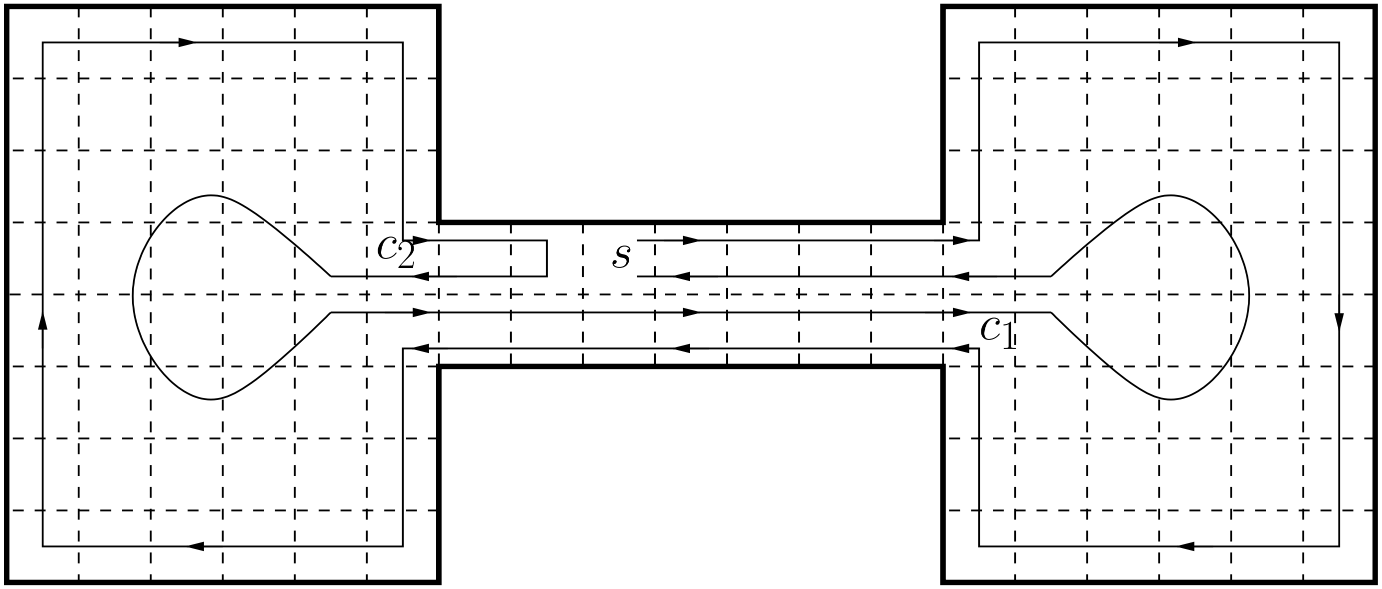
- Better than ratio 2 in fleshy environments
- Visit only the vertices
- Smart DFS!
- Number of steps: $C + \frac{1}{2}E - 3$
- $\frac{4}{3}$ competitive



Two improvement for DFS

First idea: Move along the shortest path to the next *free* cell!

Second idea: Split into different areas happens: Work on the part where the starting point is not inside! Farther away!



Smart DFS: 2. Improvement!

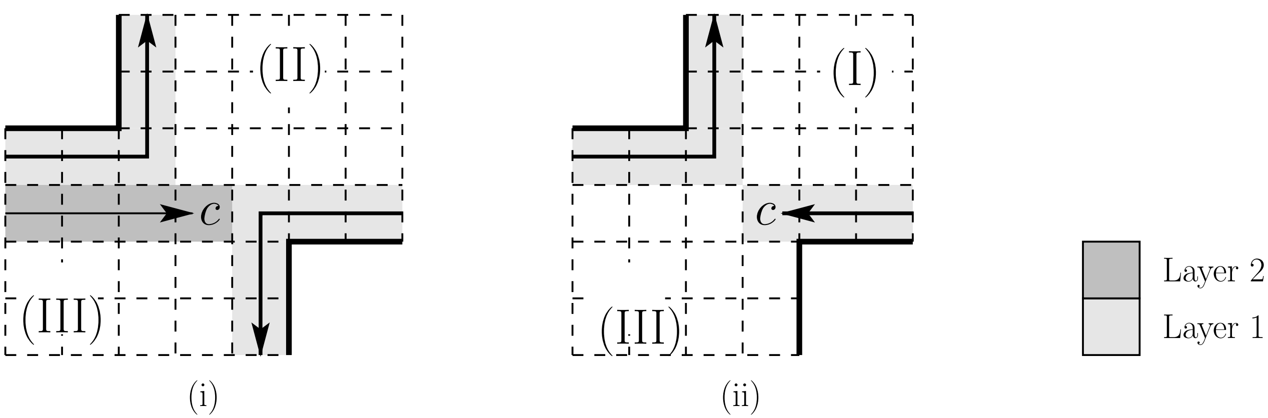
- Split-cell occurs in Layer l: How to proceed?
- Where is the starting point?

(I) Component K_i *fully* enclosed by Layer l.

(II) Component K_i *not* visited by Layer l

(III) Component K_i *partially* enclosed by Layer l.

- Visit component of type (III) last! Starting point!
- Special cases: One step Right-Hand-Rule



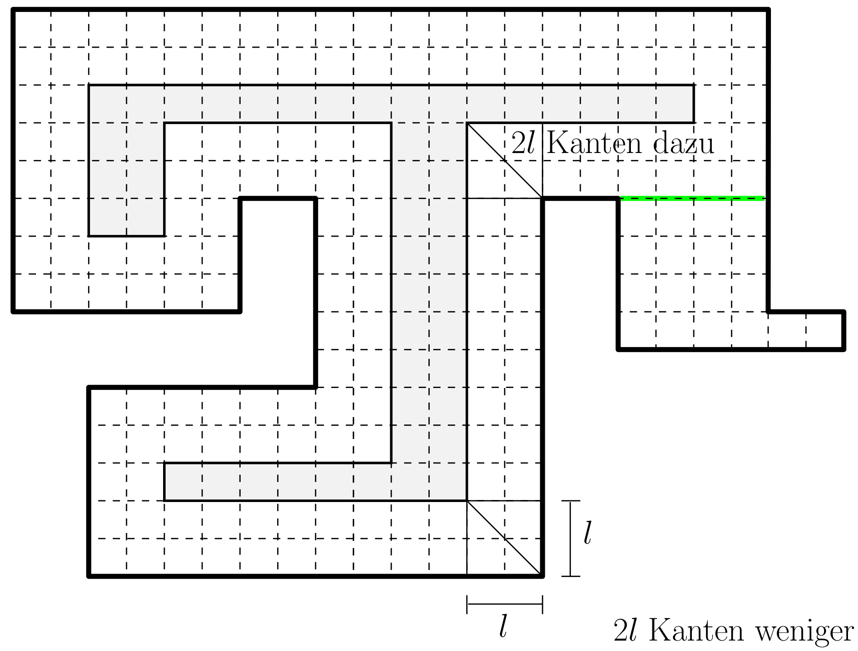
Edgelemma!

Lemma: The l-Offset of a simple gridpolygon P has *at least* $8l$ edges less than P.

$8l$ edges less

Proof: Surround the l-Offset in CW order

- Assume: Remains connected
- Left curve: l-Offset wins $2l$ edges.
- Right curve: l-Offset looses $2l$ edges.
- Altogether 4 more right curves than left curves (Turning angle 2π!)
- Disconnection improves the result
- l-Offset has at least $8l$ edges less



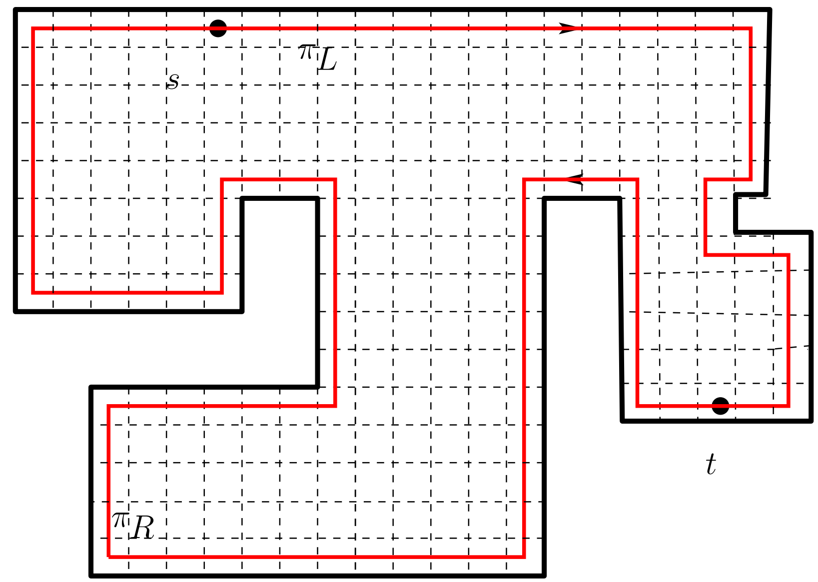
Distancelemma!

Lemma: The shortest path between to cells s and t of a simple gridpolygon P with $E(P)$edges has at most $\frac{1}{2}E(P) - 2$ steps.

Distancelemma! $\pi \leq \frac{1}{2}E(P) - 2$

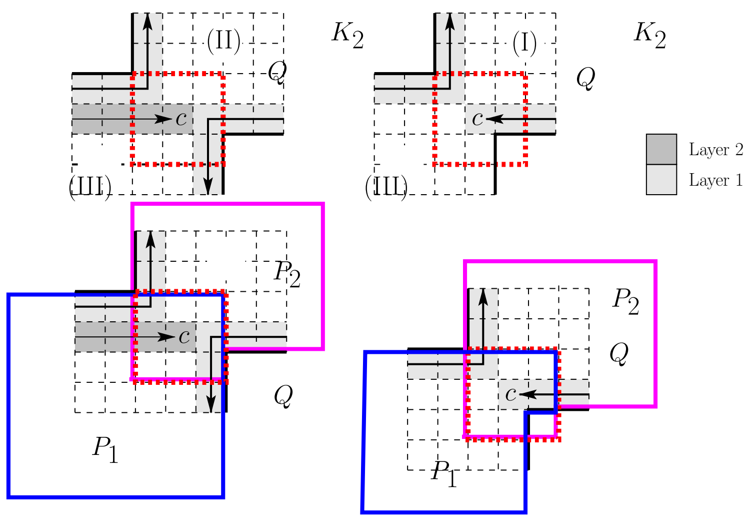
Proof:

- s and t in 1-Layer, otherwise move them to the boundary
- Along the boundary (left) π_L, (right) π_R
- Roundtrip: Count edges!
- Roundtrip: At least 4 edges more than cells/steps
- Let π be shortest patp
- $|\pi_L| + |\pi_R| = E(P) - 4 \Rightarrow \pi \leq \frac{1}{2}E(P) - 2$



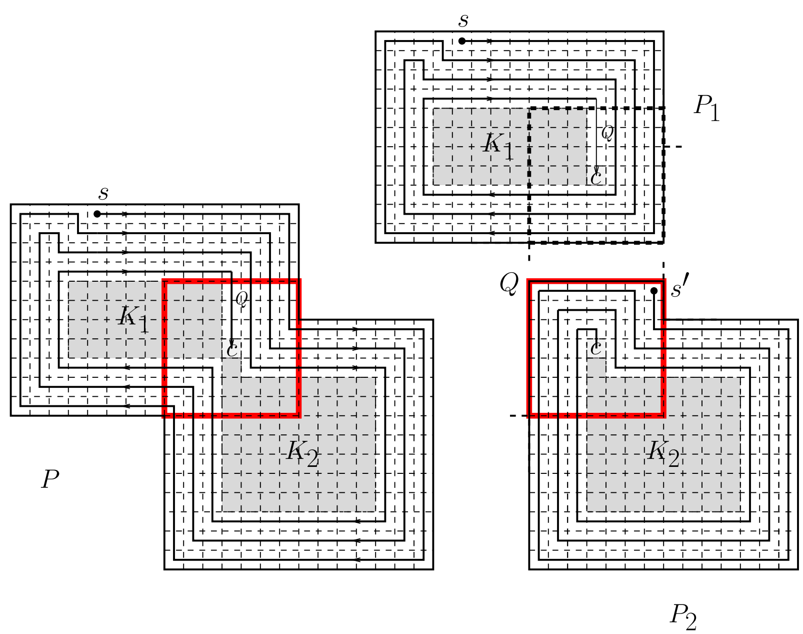
Decomposition of P at split-cell

- Decomposition Rectangle Q: $2q + 1$
- Cases: K_2 of type I) ($q = l$) or vom type II) ($q = l - 1$)
- P_2, such that $K_2 \cup \{c\}$ is the q-Offset of P_2
- $P_1 := ((P \backslash P_2) \cup Q) \cap P$ Intersection with P for the movements



Decomposition of P

- Decomposition Rectangle Q: $2q + 1$
- P_2, such that $K_2 \cup \{c\}$ is the q-Offset of P_2
- $P_1 := ((P \backslash P_2) \cup Q) \cap P$
- Path remains guilty!

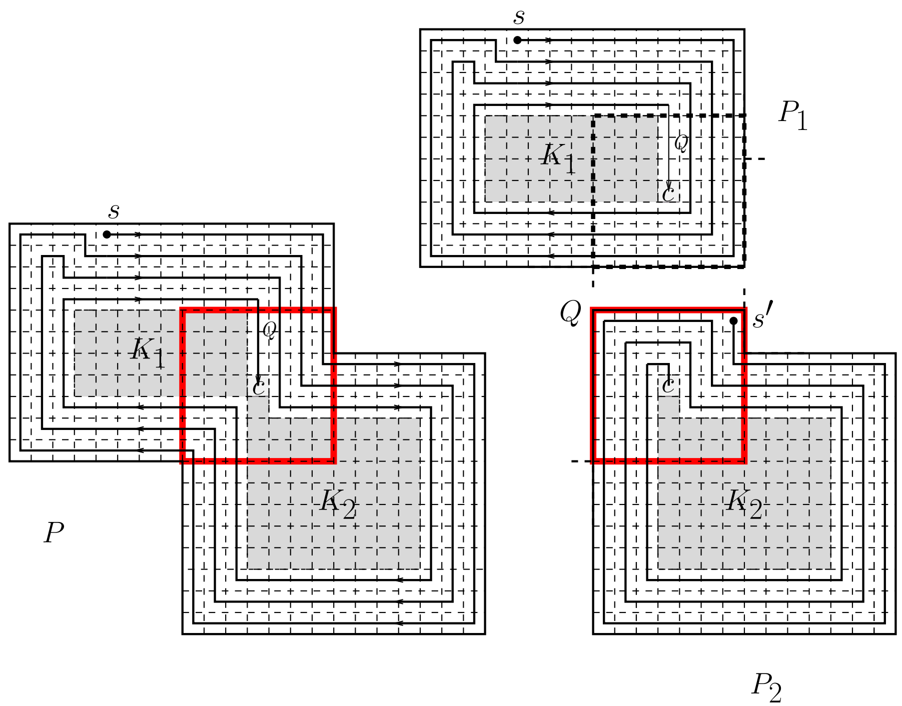


Analysis: Visity beyond the cells

- Any cell is visited once
- Number of steps $S(P)$: Visit cells (-1) plus additional visits
- $S(P) := C(P) + \mathit{excess}(P)$
- Calculate $\mathit{excess}(P)$

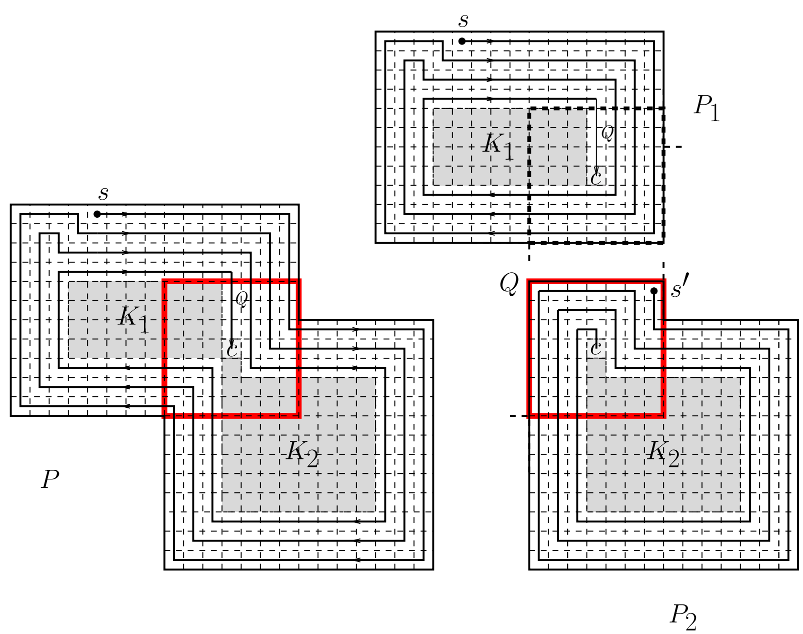
Excesslemma

Lemma: P gridpolygon and c a split-cell, such that P splits into K_1 and K_2 (for the first time). Let K_2 be the component, that is visited first. We have: *excess*$(P) \leq$ *excess*$(P_1) +$ *excess*$(K_2 \cup \{c\}) + 1$.



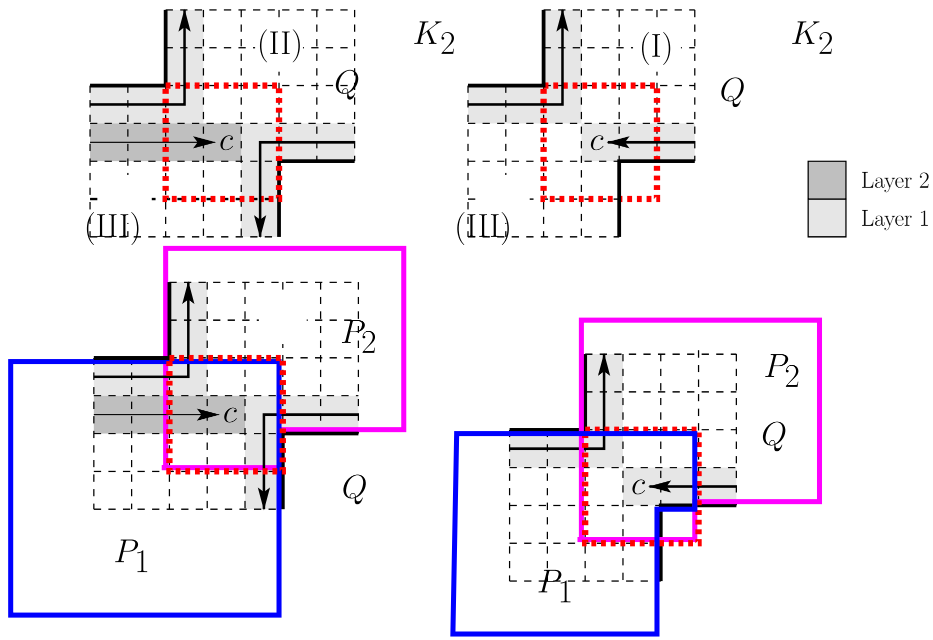
$$\mathbf{excess}(P) \le \mathbf{excess}(P_1) + \mathbf{excess}(K_2 \cup \{c\}) + 1.$$

- Explore $K_2 \cup \{c\}$ after c by SmartDFS, return to c
- Gives: max. *excess*$(K_2 \cup \{c\})$ since $P_2 \backslash (K_2 \cup \{c\})$ optimal
- c twice: plus 1
- Then move to P_1: Maximal *excess*(P_1)



Relationsship edges of P und Q Lemma 1.14

Lemma: P, P_1, P_2 und Q as given. For the number of edges we have $E(P_1) + E(P_2) = E(P) + E(Q)$.



Relationsship edges of P und Q

$E(P_1) + E(P_2) = E(P) + E(Q)$.

Two arbitrary gridpolygons P_1 und P_2 gilt:
$E(P_1) + E(P_2) = E(P_1 \cup P_2) + E(P_1 \cap P_2)$.

For $Q' := P_1 \cap P_2$, we have:

$$\begin{aligned} E(P_1) + E(P_2) &= E(P_1 \cap P_2) + E(P_1 \cup P_2) \\ &= E(Q') + E(P \cup Q) \\ &= E(Q') + E(P) + E(Q) - E(P \cap Q) \\ &= E(P) + E(Q), \text{ da } Q' = P \cap Q \end{aligned}$$

Exploration Theorem

Theorem: SmartDFS explores a simple gridpolygon P with C cells and E boundary edges with at most $C + \frac{1}{2}E - 3$ steps.

Proof: Induction over number of components

- **Induction base**: One component
- Visit cells: $C - 1$, back to start
- Shortest path Lem: $\frac{1}{2}E(P) - 2 + C - 1 = C + \frac{1}{2}E - 3$

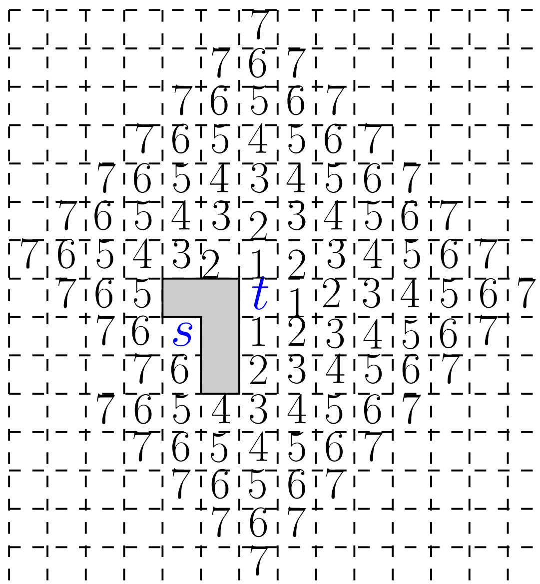
Exploration Theorem: $C + \frac{1}{2}E - 3$

- **Induction step**: split-cell c, $K_1, K_2(first), P_1, P_2, Q$
- Q with $2q+1 \times 2q+1$: Typ (I) $q = l$, Typ (II) $q = l - 1$

$$\begin{aligned}
excess(P) &\leq excess(P_1) + excess(K_2 \cup \{c\}) + 1 && \textbf{Exc. Lem.}\\
&\leq \frac{1}{2}E(P_1) - 3 + \frac{1}{2}E(K_2 \cup \{c\}) - 3 + 1 && \textbf{I.H.}\\
&\leq \frac{1}{2}E(P_1) - 3 + \frac{1}{2}(E(P_2) - 8q) - 3 + 1 && \textbf{Offset Lem.}\\
&= \frac{1}{2}(E(P_1) + E(P_2)) - 4q - 5\\
& \quad E(P_1) + E(P_2) = E(P) + 4(2q+1) && \textbf{Rel. Lem. + Def.}\\
&= \frac{1}{2}E(P) - 3
\end{aligned}$$

Shortest path over eplored cells

- Offline Problem, within SmartDFS
- Wave front from t to s, label with L_1 distance to t
- Mark adjacent cell with label $+ 1$, Queue

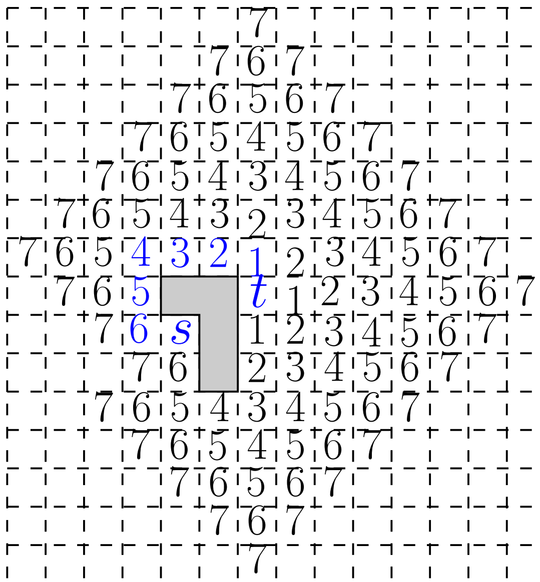


Algorithm of Lee: Labels

Datenstruktur: Queue Q
{Initialise:}
Q.InsertItem(t);
Markiere t mit 0;
{Wave Propagation:}
loop
    $c := Q.\text{RemoveItem}()$;
    for all Cells x such that x adjacent to c and x not marked **do**
        Mark x with label $label(c) + 1$;
        Q.InsertItem(x);
        if $x = s$ **then** break loop;
    end for
end loop

Algorithm of Lee: Compute the path

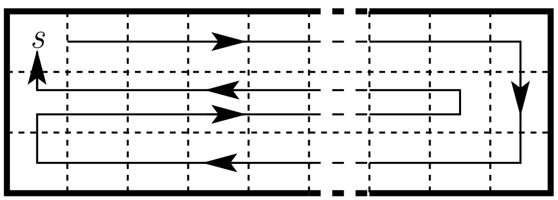
Move along cells with decreasing labels starting from s nach t.



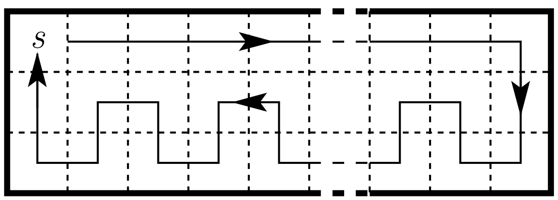
Not unique!

Competitive Ratio: SmartDFS

- Compare optimal path vs. SmartDFS
- THIS is the worst-case
- $S(P) \leq \frac{4}{3} C(P) - 2$
- $3 \times m$, m even, exact! I.e., 30 against 24!



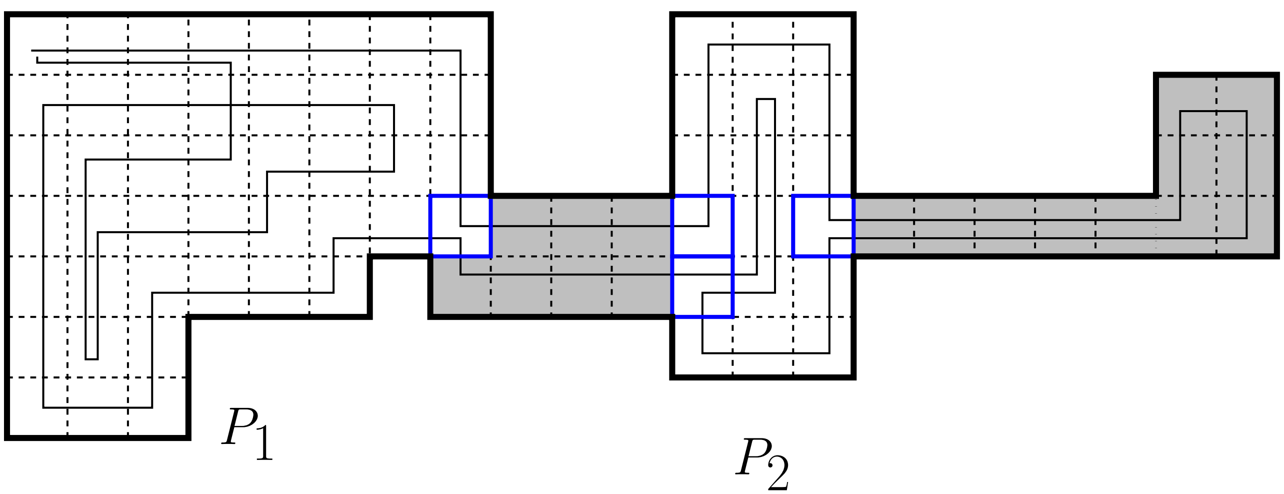
SmartDFS



Optimale Strategie

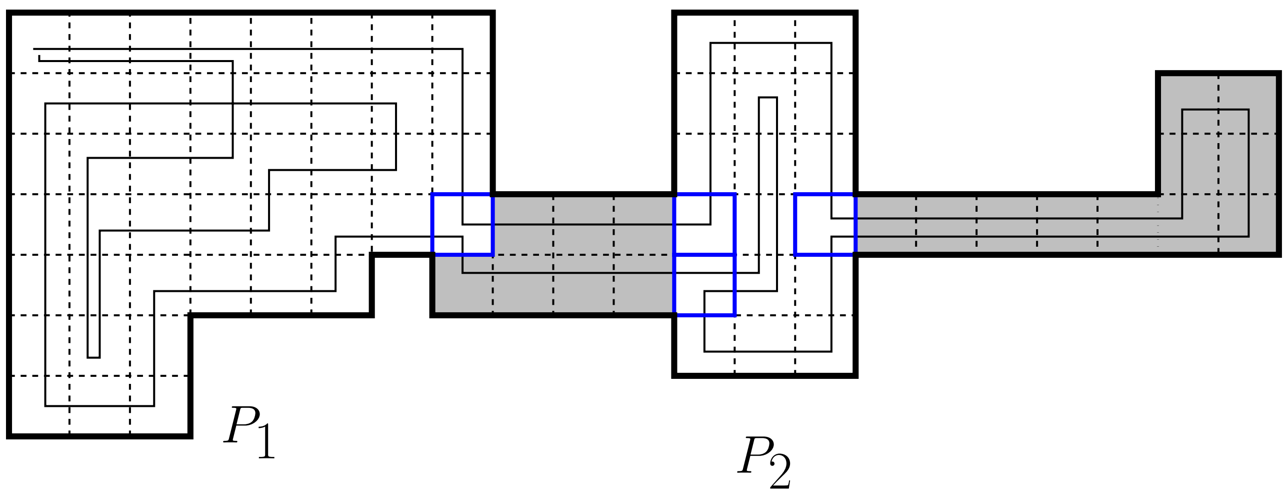
Structural properties: SmartDFS

- Optimal in *corridors* of width 1/2
- Cells that do not change the layer-number of the neighbors, if we delete them!
- Neighbours of layer 1! Gates for the corridors!
- **Def.** Narrow passages!



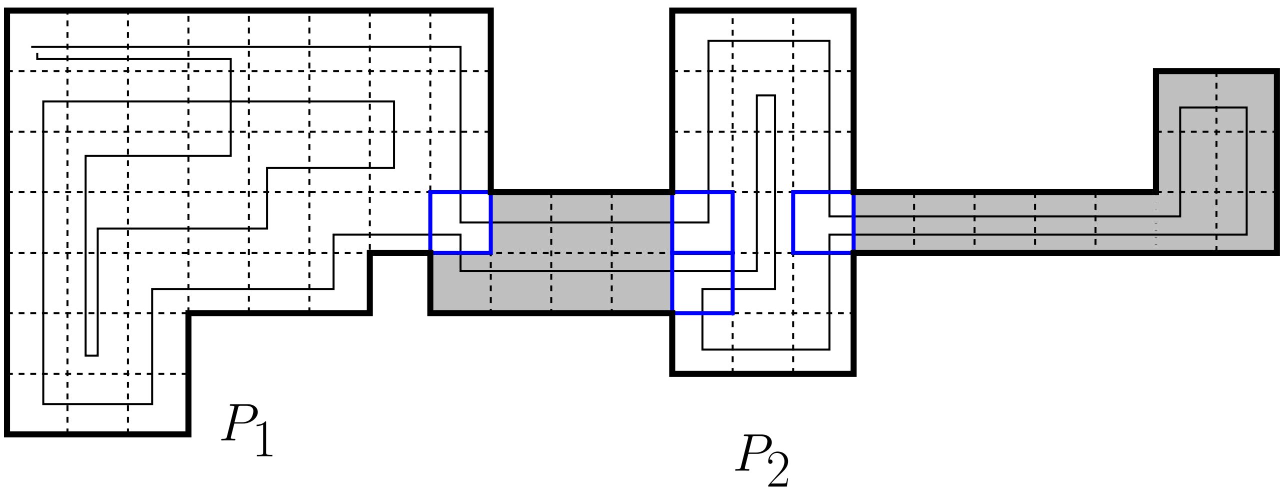
Omit narrow passages

- Omit *narrow passages*, analyse the remaining parts
- Results in sequence of polygons P_i, $i = 1, \ldots, k$
- Example P_1 und P_2: Analysis with the cells!!



Omit narrow passages

- Analyse Polygone without *narrow passages*
- Inductively over the split-cells in ther first layer
- Induction base: No split-cell in the first layer!
- Analyse such polygons first!



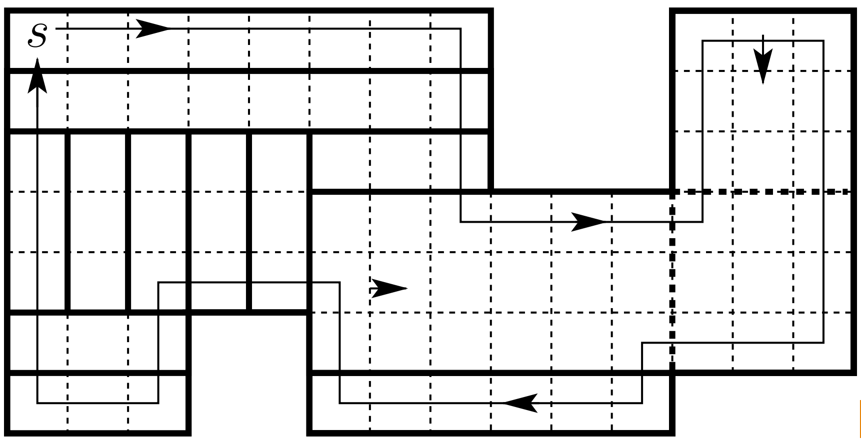
Polygons without *narrow passages* and without 1-Layer split

Lemma: P simple gridpolygon, no narrow passages, no split in the first layer. $E(P) \leq \frac{2}{3}C(P) + 6$ holds Proof:

- Exactly for 3×3 gridpolygons
- 9 cells, 12 edges
- Reduce any such gridpolygon to this base case
- Subtraction of at least 3 cells and at most 2 edges
- Properties survive

Proof $E(P) \leq \frac{2}{3}C(P) + 6$

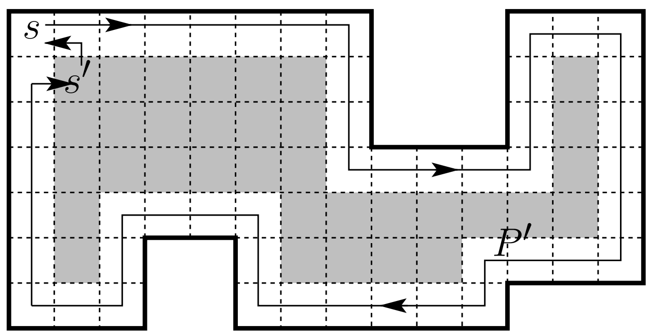
- Sequence of consistent removes of rows and column
- Subtraction of at least 3 cells and at most 2 edges
- Start with $E(P) = \frac{2}{3}C(P) + 6$
- Backward Analysis: plus (3+X) cells, plus (2-X) edges



Polygons without *narrow passages* and without 1-Layer split

Lemma: SmartDFS requires $S(P) \leq C(P) + \frac{1}{2}E(P) - 5$!

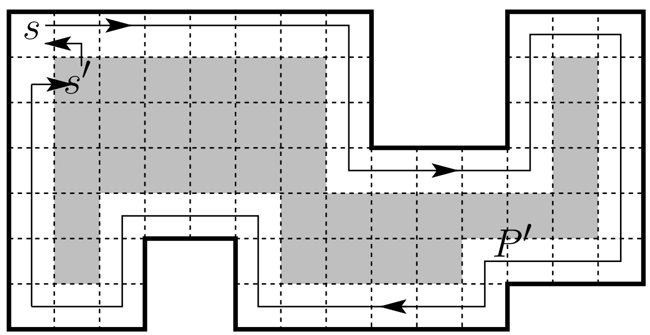
- Theorem: $S(P) \leq C(P) + \frac{1}{2}E(P) - 3$
- By assumption: SmartDFS full first round, C' steps (1-Layer)
- SmartDFS starts at s' in P'



Polygons without *narrow passages* and without 1-Layer split

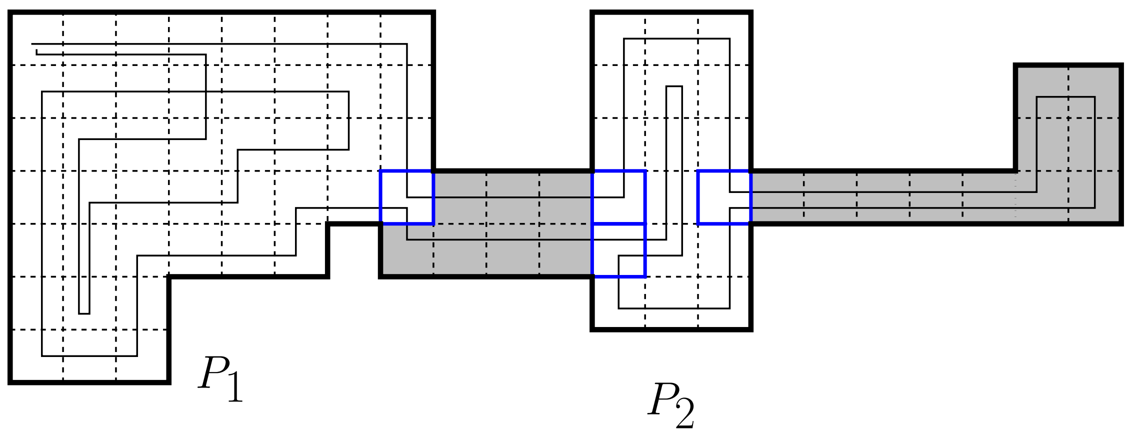
Lemma: SmartDFS requires $S(P) \leq C(P) + \frac{1}{2}E(P) - 5$!

- P' has exact 8 egdes less, by Offset Lemma
- 2 steps back to s, final step!
- $S(P) \leq C(P) + \frac{1}{2}(E(P) - 8) - 3 + 2 = C(P) + \frac{1}{2}E(P) - 5$



Theorem: SmartDFS ist $\frac{4}{3}$ kompetitiv

- Narrow passages optimal, sequence of P_i independently!
- Only cells and steps, no edges!!
- Induction in P_i over split-cell number! $S(P_i) \leq \frac{4}{3}C(P_i) - 2$
- Induktion base: Use special lemmata!



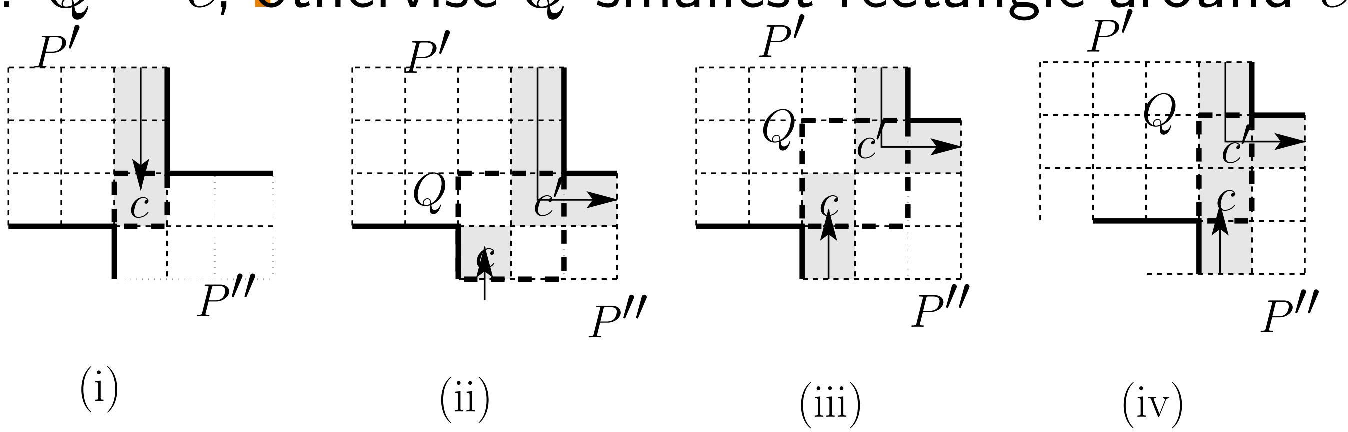
Induction base: $S(P_i) \leq \frac{4}{3}C(P_i) - 2$

- P_i no split-cell means, no split-cell in Layer 1
- Apply case-sensitive Lemma: $C(P) + \frac{1}{2}E(P) - 5$
- Apply structural Lemma: $E(P) \leq \frac{2}{3}C(P) + 6$

$$
\begin{aligned}
S(P_i) &\leq C(P_i) + \frac{1}{2}E(P_i) - 5 \\
&\leq C(P_i) + \frac{1}{2}\left(\frac{2}{3}C(P_i) + 6\right) - 5 \\
&= \frac{4}{3}C(P_i) - 2
\end{aligned}
$$

Induktion step: $S(P_i) \leq \frac{4}{3}C(P_i) - 2$

- Split-cell in first layer of P_i, otherwise done: Two Cases
- Split by c adjacent to some c'
- Typ (I) (curr. layer not) or Typ (II) (curr. layer fully.) component
- Split into P' and P'' with Rectangle/Square Q
- Case (i): $Q = c$, othervise Q smallest rectangle around c,c'

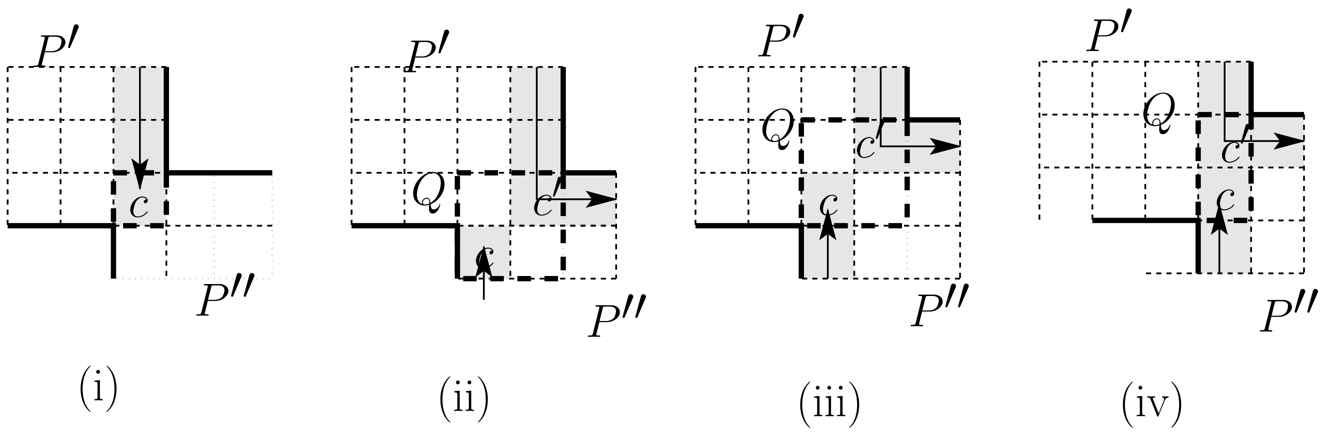


Case (i): $S(P_i) \leq \frac{4}{3}C(P_i) - 2$

- $S(P_i) = S(P') + S(P'')$ (Gate) $C(P_i) = C(P') + C(P'') - 1$
- Induction: For P' and P'' (less split-cells)

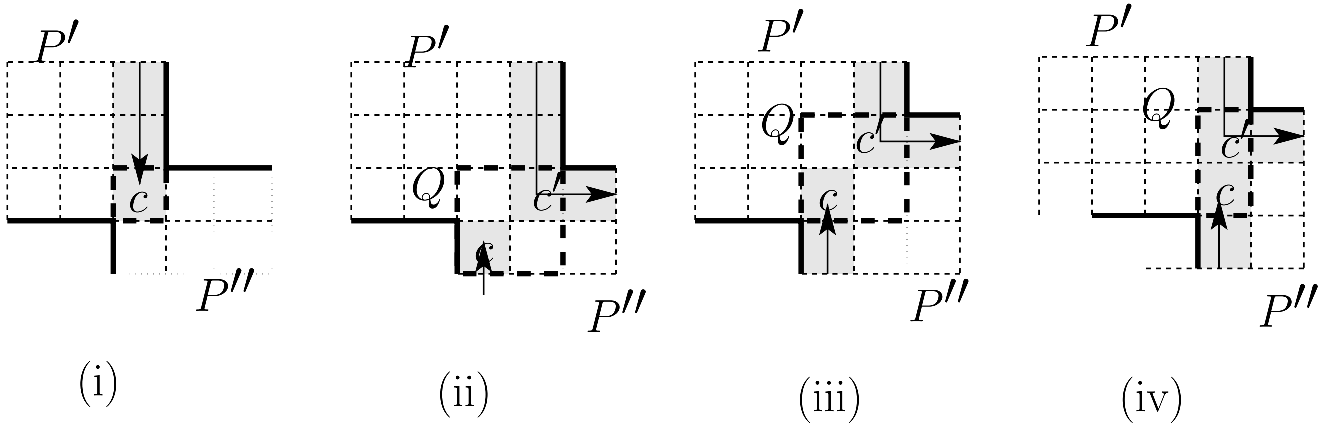
$$S(P_i) = S(P') + S(P'') \leq \frac{4}{3}C(P') - 2 + \frac{4}{3}C(P'') - 2$$

$$\leq \frac{4}{3}C(P_i) + \frac{4}{3} - 4 < \frac{4}{3}C(P_i) - 2$$



Fall (ii),(iii): $S(P_i) \leq \frac{4}{3}C(P_i) - 2$

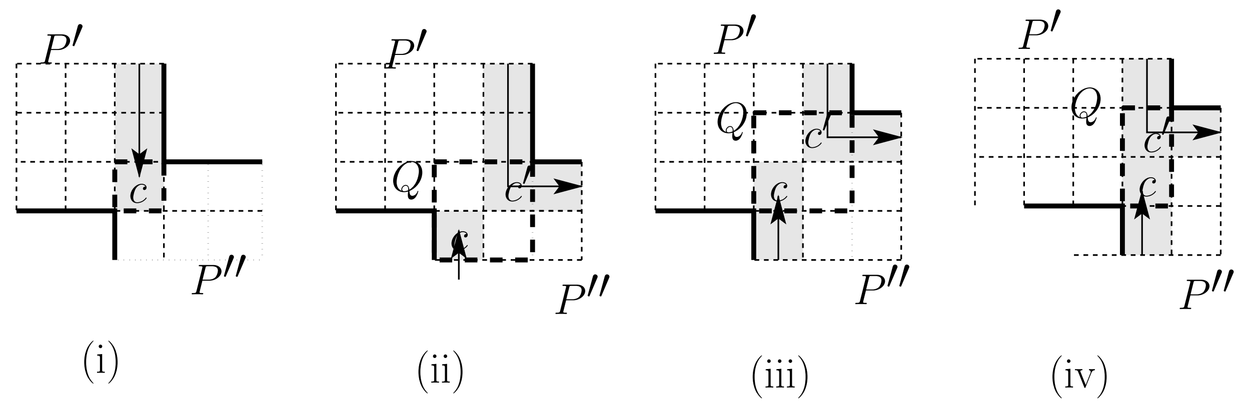
- $|Q| = 4$ but save 4 steps!
- P', P'' separately (I.H.) but
- Path in P_i from c' to c or from c to c' done in P',P''
- Save at least 4=$|Q|$ steps



Fall (ii),(iii): $S(P_i) \leq \frac{4}{3}C(P_i) - 2$

- At least 4=$|Q|$ steps less
- $S(P_i) = S(P') + S(P'') - 4$ and $C(P_i) = C(P') + C(P'') - 4$
- Apply I.H. for P' and P''

$$\begin{aligned} S(P_i) &= S(P') + S(P'') - 4 \leq \frac{4}{3}C(P') + \frac{4}{3}C(P'') - 8 \\ &\leq \frac{4}{3}(C(P') + C(P'') - 4) - \frac{8}{3} < \frac{4}{3}C(P_i) - 2 \end{aligned}$$



Summary SmartDFS

- Gridpolygons without holes
- Lower bound: $\frac{7}{6}$
- SmartDFS: $\frac{4}{3}$
- More sophisticated approach: approx. $\frac{5}{4}$
- Lower bound: $\frac{20}{17}$
- Optimale Offline Solution?