Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Geometric Firefighting Plane

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December 15th, 2015

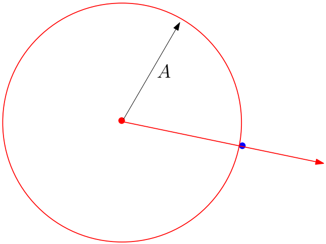
Geometric firefigthing in the plane

- Expanding fire in the plane
- Barrier curve with speed $v > 1$
- Current point outside the fire

Geometric Firefighter Problem in the plane
Instance: Expanding fire-circle spreads with unit speed from a given starting point s, start radius A.
Question: How fast must a firefighter be, to build a barrier that finally fully encloses and stops the expanding fire?

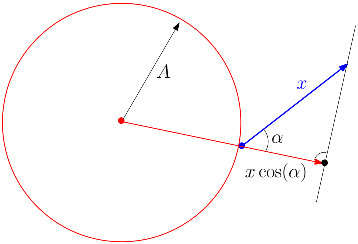
Spiral movements for speed v

Spiral movements for speed v



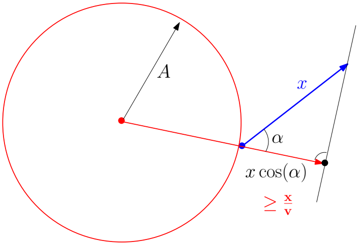
- Start on the boundary, speed $v > 1$

Spiral movements for speed v



- Start on the boundary, speed $v > 1$
- Allowed angle?

Spiral movements for speed v



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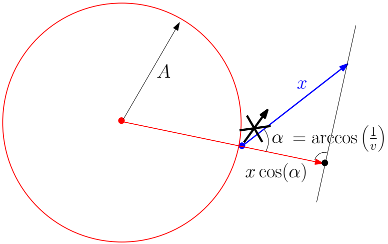
Spiral movements for speed v



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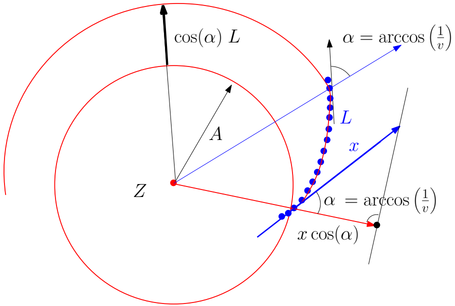
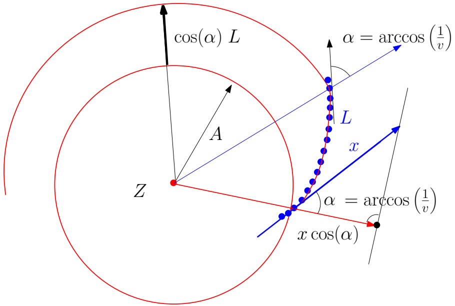
- Allowed angle?

Spiral movements for speed v



- Start on the boundary, speed $v > 1$
- Allowed angle?
- *Riding* the fire

Spiral movements for speed v



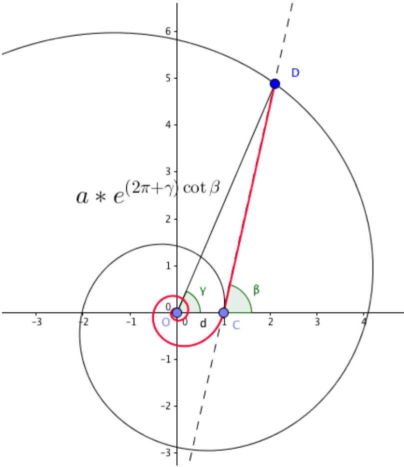
- Start on the boundary, speed $v > 1$
- Allowed angle?
- *Riding* the fire
- Log. Spiral around Z
- Excentricity α $\cos(\alpha) = \frac{1}{v}$

Properties of a spiral

- Polar coordinates $S(\varphi) := (\varphi, a \cdot e^{\varphi \cot \alpha})$
- Constant a
- $\alpha \in (0, \pi/2)$, $\cot \alpha$ from 0 to ∞
- $|S_q^p| = \frac{1}{\cos \alpha} (|Bq| - |Bp|)$

Spiralling strategy, upper bound on the speed

Bresson et al. 2008



- Spiral was constructed
- Let the fire expand
- Follows to current point D
- Speed difference?
- $\gamma(\beta) =: \gamma$
- $\overline{OC} = a$

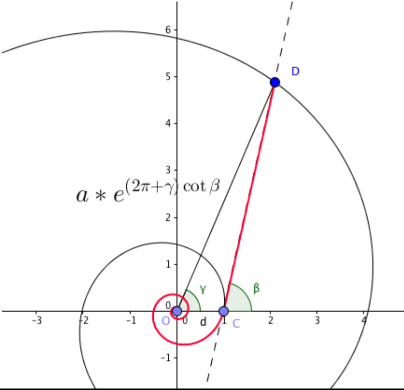
$$\frac{a \cdot e^{(2\pi+\gamma)\cot\beta}}{\sin\beta} = \frac{a}{\sin(\beta-\gamma)} \iff e^{(2\pi+\gamma)\cot\beta} = \frac{\sin\beta}{\sin(\beta-\gamma)}.$$

Spiralling strategy, upper bound on the speed

$$f(\beta) := \frac{|\text{Length spiral from } O \text{ to } D|}{|\text{Length spiral from } O \text{ to } C| + |\text{Length segment } CD|}.$$

$$f(\beta) = \frac{\frac{1}{\cos\beta} e^{(2\pi+\gamma)\cot\beta}}{\frac{1}{\cos\beta} + \frac{\sin\gamma}{\sin\beta} e^{(2\pi+\gamma)\cot\beta}} = \frac{\frac{1}{\cos\beta}\frac{\sin\beta}{\sin(\beta-\gamma)}}{\frac{1}{\cos\beta} + \frac{\sin\gamma}{\sin\beta}\frac{\sin\beta}{\sin(\beta-\gamma)}} = \frac{1}{\cos\gamma}.$$

- $CD = a\frac{\sin\gamma}{\sin\beta} e^{(2\pi+\gamma)\cot\beta}$
- $\frac{a}{\cos\beta} e^{(2\pi+\gamma)\cot\beta}$
- $\frac{a}{\cos\beta}$

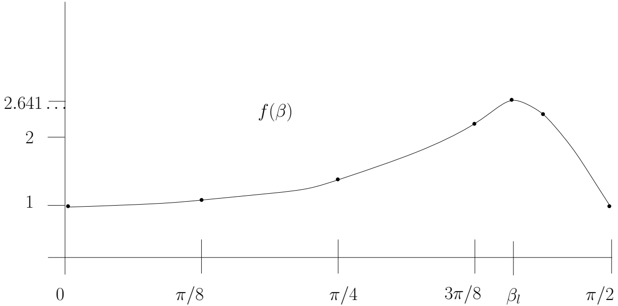


Properties of $f(\beta)$

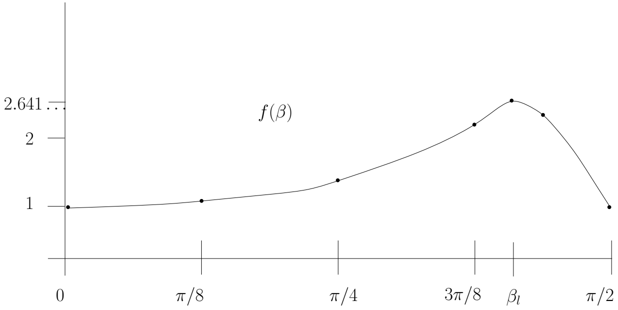
$$f(\beta) = \frac{\frac{1}{\cos\beta} e^{(2\pi+\gamma)\cot\beta}}{\frac{1}{\cos\beta} + \frac{\sin\gamma}{\sin\beta} e^{(2\pi+\gamma)\cot\beta}} = \frac{1}{\cos\gamma}\,.$$

$$\lim_{\beta\mapsto 0} f(\beta) = \lim_{\gamma\mapsto 0} \frac{1}{\cos\gamma} = 1\,.$$

$$\limsup_{\beta\mapsto\pi/2\nearrow} f(\beta) \le \limsup_{\beta\mapsto\pi/2\nearrow} e^{(2\pi+\gamma)\cot\beta} \le \lim_{\beta\mapsto\pi/2\nearrow} e^{(5\pi/2)\cot\beta} = 1\,.$$



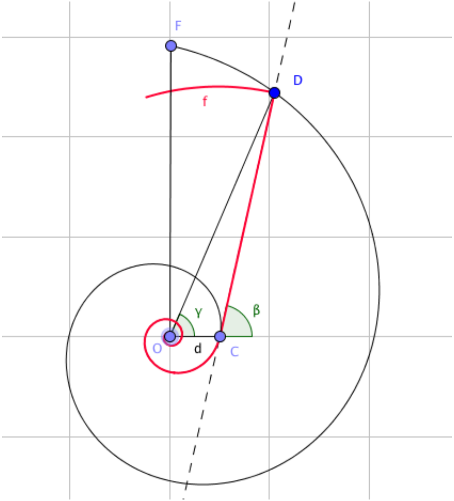
Properties of a spiral



- Maps to 1 at the boundary
- $\gamma(\beta)$ continuous, well-defined, f continuous, well-defined
- Unique global maximum: $v_l := \max_{\beta \in (0,\pi/2)} f(\beta)$.
- Numerically: $\beta_l = 1.29783410242\ldots$ and gives $v_l = f(\beta_l) = 2.614430844\ldots$ and $\gamma(\beta_l) = 1.178303978\ldots$

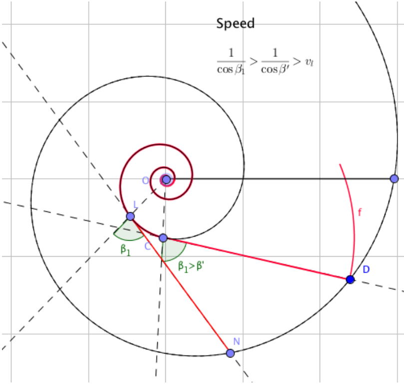
Construct strategy with speed $v > v_l = 2.614430844\ldots$

- For any speed $v > v_l$ spiral keeps in front of fire
- Use $v_1 = \frac{1}{\cos \beta_1} > v_l$ and spiral with excentricity β_1



Construct strategy with speed $v > v_l = 2.614430844\ldots$

- Use $v_1 = \frac{1}{\cos\beta_1} > v_l$ and spiral with excentricity β_1
- Make it a legal start spiral: β_1 helps for starting!!!
- Starting circle construction:

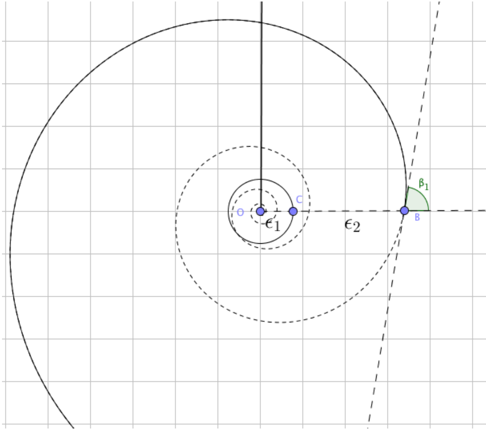


- $v' = \frac{1}{\cos\beta'}$ with $v_1 > v' > v_l$
- F is met $t_1 = t\cos\beta_1$,
  $t_2 = t\cos\beta' > t_1$
- $x = t(\frac{1}{\cos\beta_1} - \frac{1}{\cos\beta'})$ time from N to D for the fire

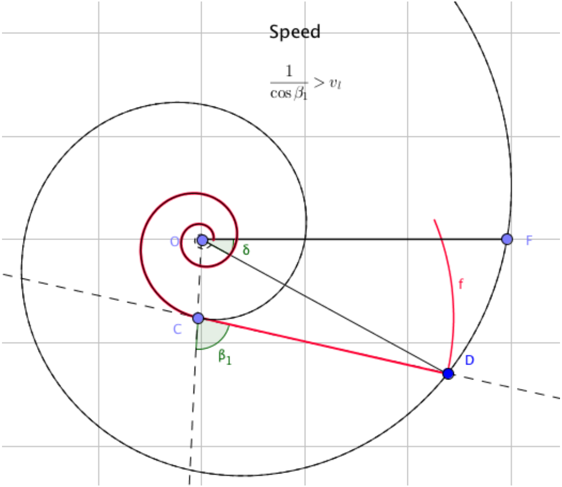
Construct strategy with speed $v > v_l = 2.614430844\ldots$

- $x = t(\frac{1}{\cos\beta_1} - \frac{1}{\cos\beta'})$ time from N to D
- Use x for the start

- $\frac{1}{\cos\beta'}(\epsilon_1 + \epsilon_2) - \epsilon_2 < x$
- Speed $v_1 = \frac{1}{\cos\beta_1} > v' > v_l$ helps
- Angle β_1 helps!

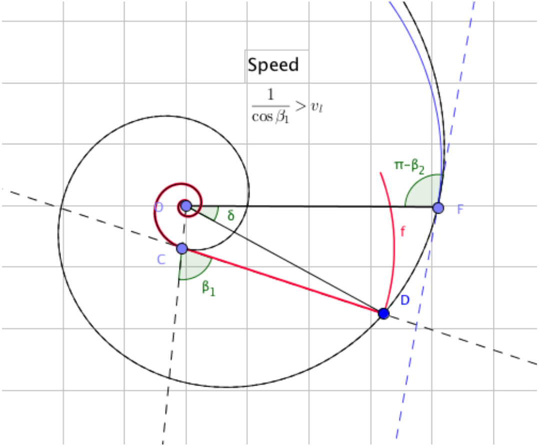


- For any $v_1 = \frac{1}{\cos \beta_1} > v_l$
- Admissable spiral, starting radius $C_1 = (\epsilon_1 + \epsilon_2)$, excentricity β_1



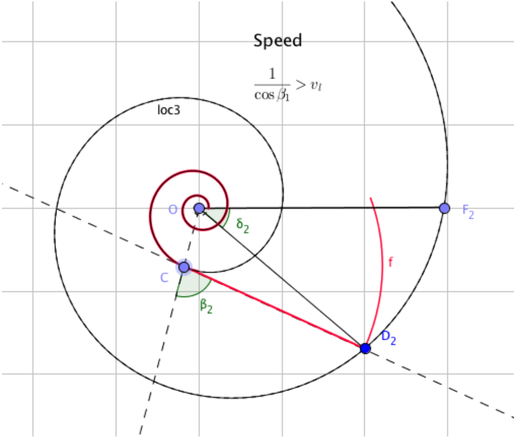
2. Enclosement by iterations

- $F = (e^{2k_1\pi \cot \beta_1}, 0)$ with excentricity $\beta_2 > \beta_1$ and starting radius $C_2 = C_1 e^{2k_1\pi \cot \beta_1}$
- Admissable, if $\beta_2 > \beta_1$ close to β_1.



2. Iteration

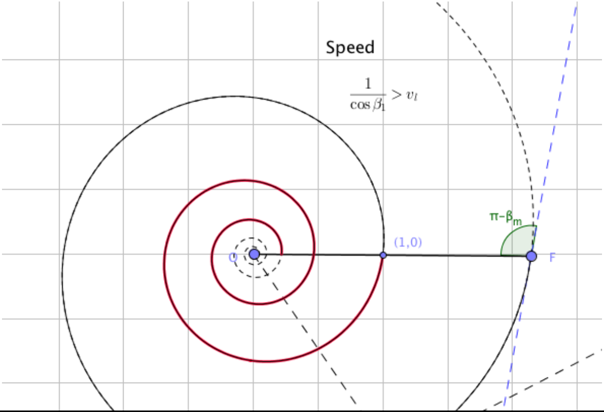
- Spiral with β_2 until angle $2k_2\pi$
- F_2 and the fire is behind at D_2
- $\beta_1 < \beta_2 < \ldots < \beta_l$ and spirals $C_i e^{2\pi k_i \cot \beta_i}$



2. Many iterations $\beta_m > \beta_{m-1} > \cdots > \beta_1$

$$\frac{1}{\cos\beta_m} > \frac{1}{\cos\beta_1}\left(\frac{1}{\cos\beta_m}e^{2\pi\cot\beta_m} + (e^{2\pi\cot\beta_m} - 1)\right)$$

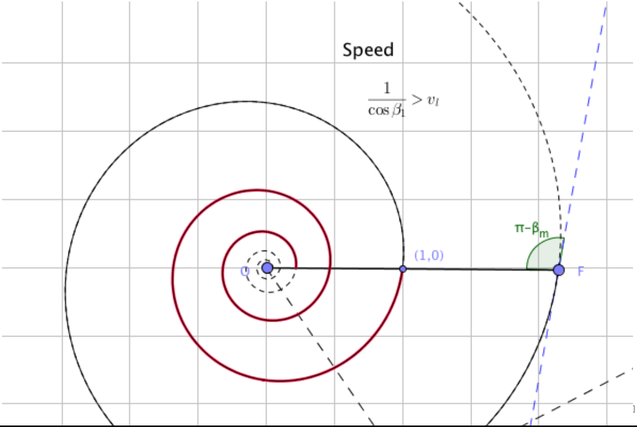
- $\frac{1}{\cos\beta_m}e^{2\pi\cot\beta_m}$ and $e^{2\pi\cot\beta_m} - 1$
- Versus $\frac{1}{\cos\beta_m}$ (scaling!)



2. Many iterations $\beta_m > \beta_{m-1} > \cdots > \beta_1$

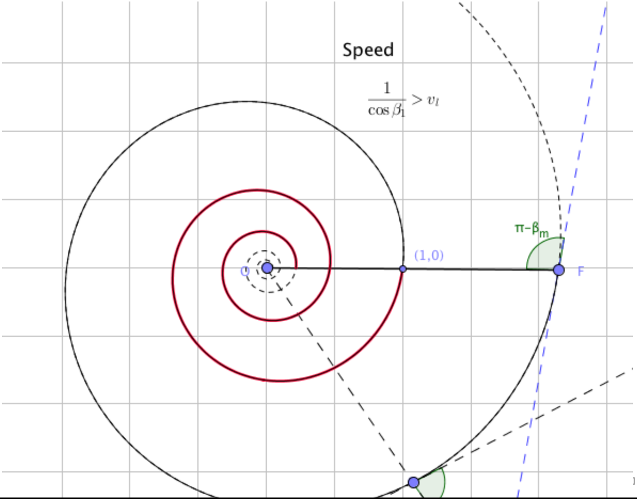
$$\frac{1}{\cos\beta_m} > \frac{1}{\cos\beta_1}\left(\frac{1}{\cos\beta_m} e^{2\pi\cot\beta_m} + (e^{2\pi\cot\beta_m} - 1)\right)$$

- Example. $\beta_1 \approx 1.191388\ldots$ and $\frac{1}{\cos\beta_1} = 2.7$ we require $\beta_m > 1.4268$.



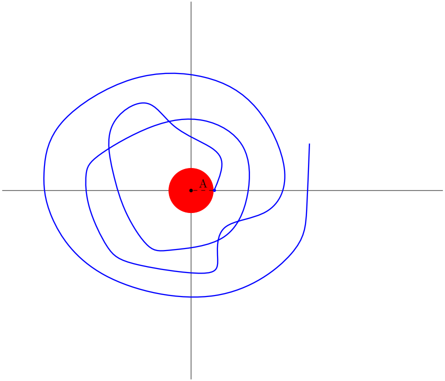
2. Many iterations $\beta_m > \beta_{m-1} > \cdots > \beta_1$

Theorem 56: (Bresson et al. 2008) For any speed $v > v_l \approx 2.614430844$ there is a spiralling strategy that finally encloses an expanding circle that expands with unit speed.



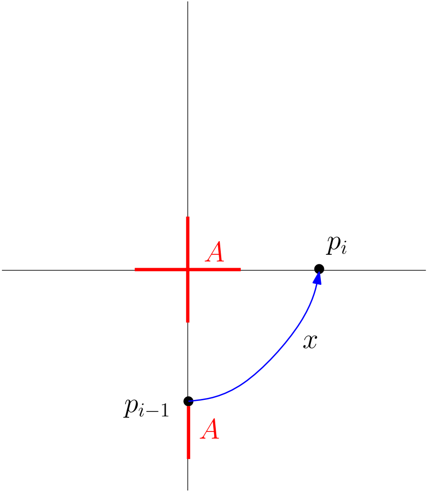
Lower bound construction, spiralling strategies!

- Start at the fire!
- Spiralling strategies!
- Visit four axes in cyclic order
- Visit axes in increasing distance



Theorem 58: Each "spiralling" strategy must have speed $v > 1.618\ldots$ (golden ratio) to be successful.

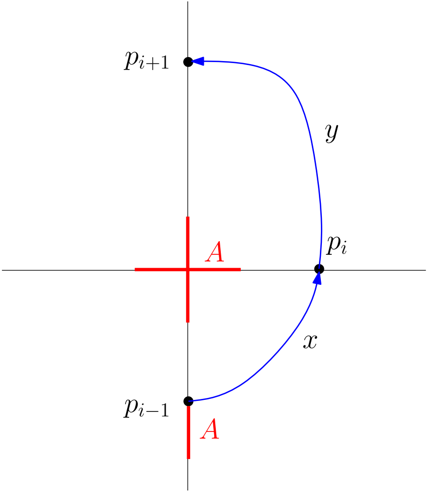
Proof of lower speed bound: suppose $v \leq 1.618$



By induction:
On reaching p_i,
interval of length A below
p_{i-1} is on fire.

(Induction base!)

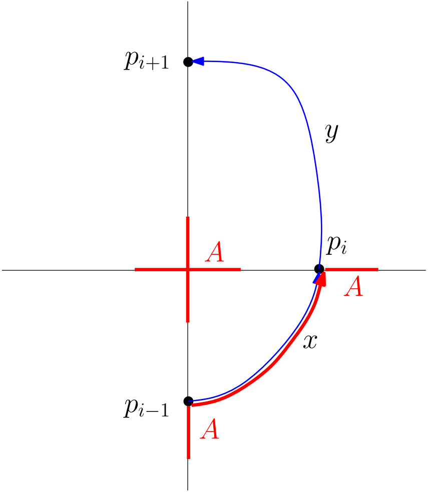
Proof of lower speed bound: suppose $v \leq 1.618$



Inductive Step:

After arriving p_{i+1}
fire moves at least $x + A$

Proof of lower speed bound: suppose $v \leq 1.618$



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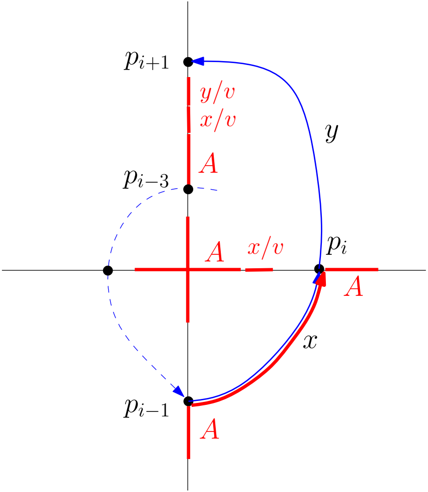

On reaching p_{i+1}:

1. $A + \frac{x}{v} \leq p_i \leq x$ and
2. $A + \frac{x}{v} + \frac{y}{v} \leq p_{i+1} \leq y$

$\Longrightarrow \frac{1}{v(v-1)} x + \frac{1}{v-1} A \leq \frac{y}{v}$

$\Longrightarrow x + A \leq \frac{y}{v}$

from $v^2 - v \leq 1$