Voronoi Diagram and Delaunay Triangulation

Randomized Incremental Construction

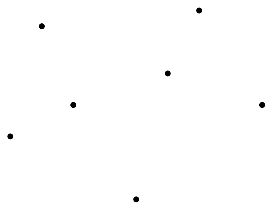
Chih-Hung Liu

May 13, 2015

Outline

Voronoi Diagram

- Given a set S of n point sites, Voronoi Diagram $V(S)$ is a planar subdivision



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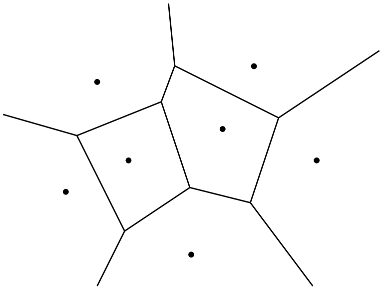

Voronoi Diagram

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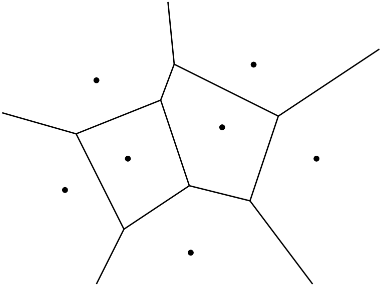
  1. Each region contains exactly one site $p \in S$ and is denoted by $\mathrm{VR}(p, S)$.



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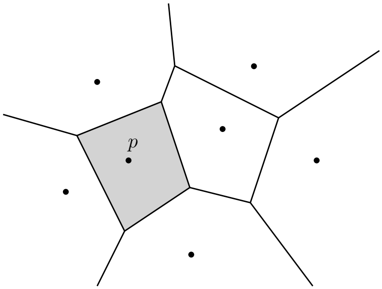

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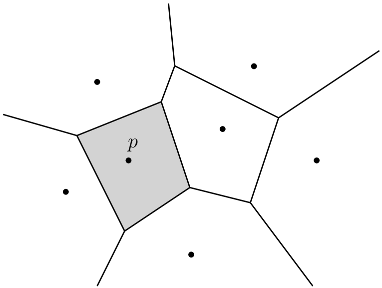

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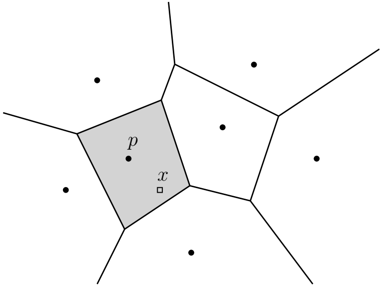

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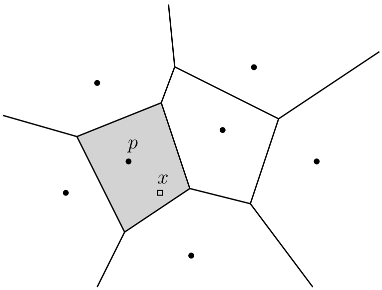
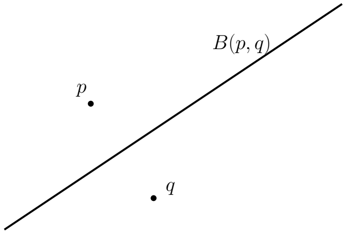
  1. Each region contains exactly one site $p \in S$ and is denoted by $\mathrm{VR}(p, S)$.
  2. For each point $x \in \mathrm{VR}(p, S)$, p is its closest site in S.
- $\mathrm{VR}(p, S)$ is the locus of points closer to p than any other site.



Voronoi Region

- Bisector $\boldsymbol{B(p,q)} = \{x \in R^2 \mid d(x,p) = d(x,q)\}$.



Voronoi Region

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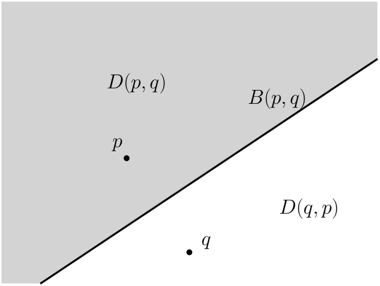
  - Two half-planes $D(p,q)$ and $D(q,p)$ separated by $B(p,q)$.



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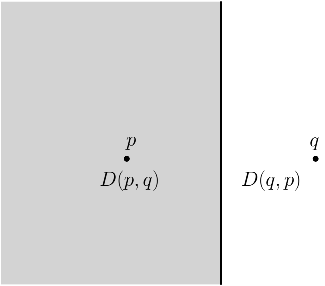

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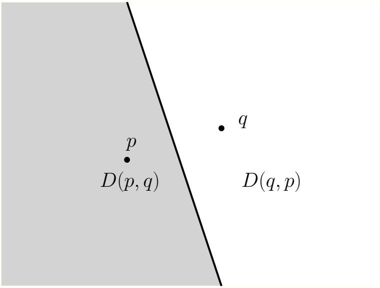

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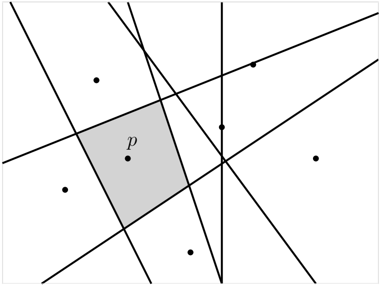
  - Two half-planes $D(p,q)$ and $D(q,p)$ separated by $B(p,q)$.
- $$\mathrm{VR}(p,S) = \bigcap_{q \in S, q \neq p} D(p,q).$$



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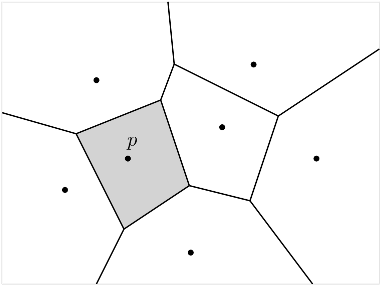
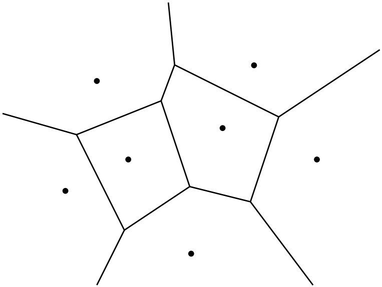


Voronoi Edge and Vertex

- Voronoi Edge
  - Common intersection between two adjacent Voronoi regions $VR(p, S)$ and $VR(q, S)$



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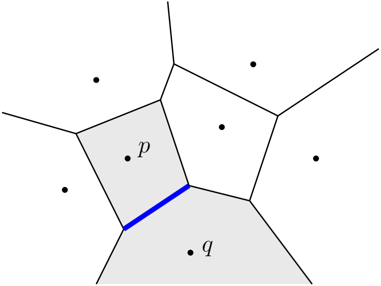

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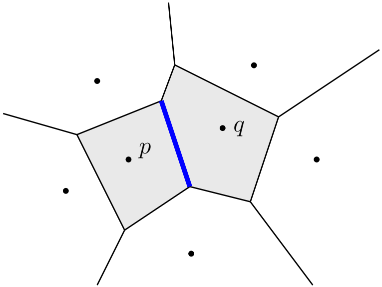

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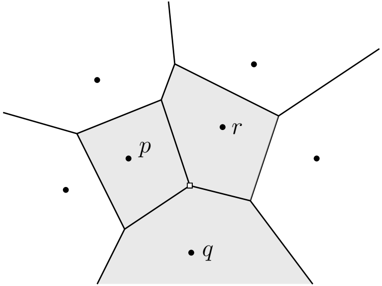
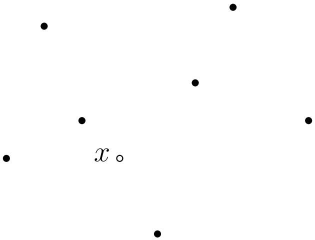
- Voronoi Vertex
  - Common intersection among more than two Voronoi regions $\mathrm{VR}(p, S)$, $\mathrm{VR}(q, S)$, $\mathrm{VR}(r, S)$, and so on.



Growing Circle

- Grow a circle from a point *x* on the plane



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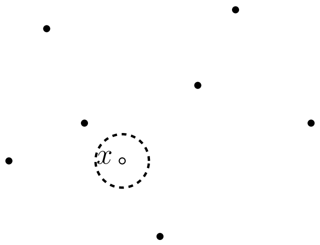

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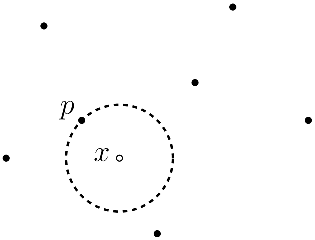

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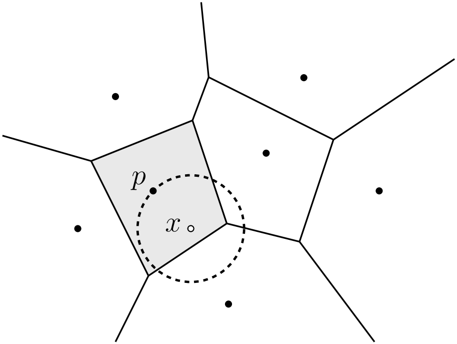

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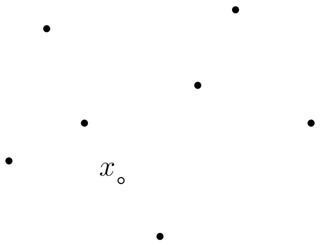

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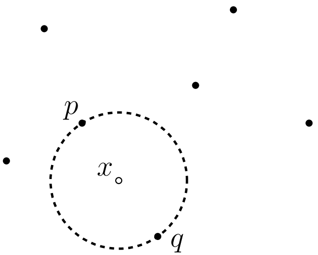

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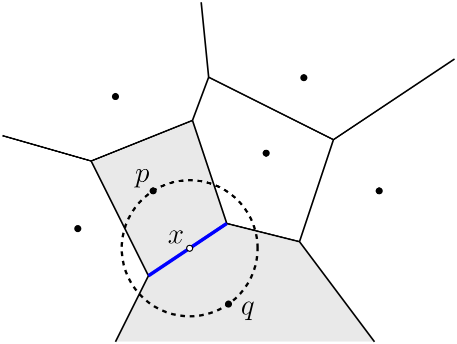

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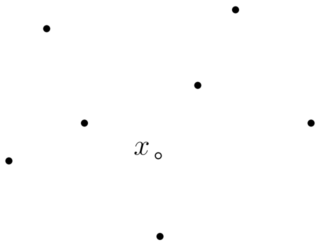

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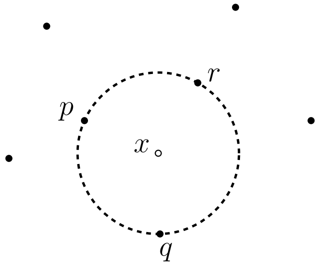

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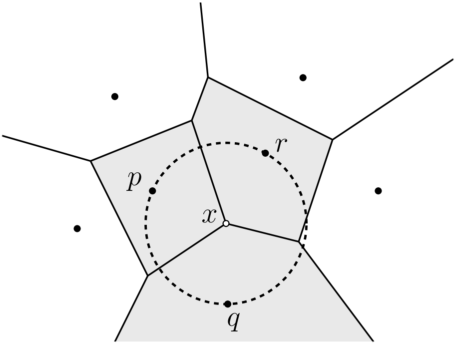
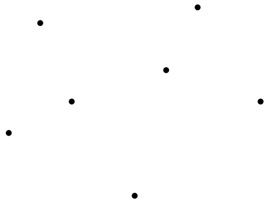


Wavefront Model (Growth Model)

- Grow circles from $\forall p \in S$ at unit speed



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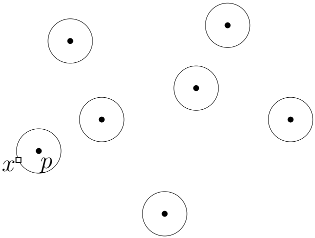

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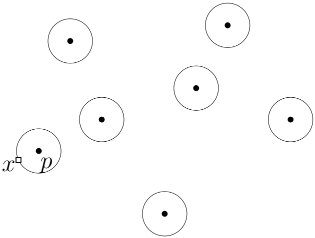

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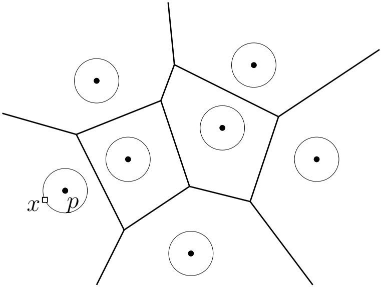

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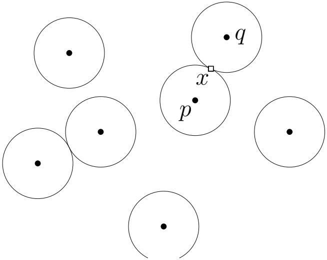

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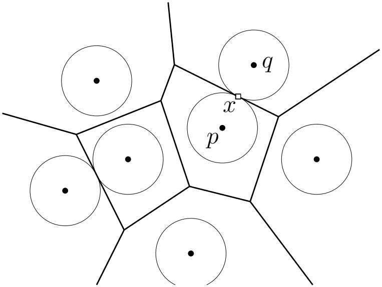

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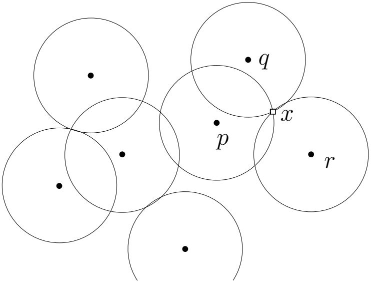

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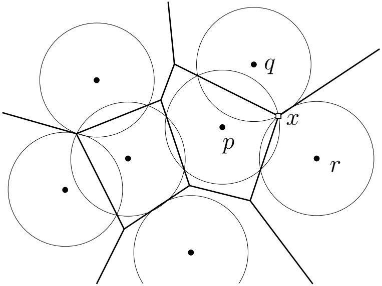
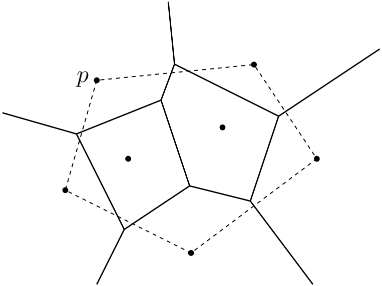


Unbounded Region

- VR(p, S) is unbounded if and only if p is a vertex of the convex hull of S.



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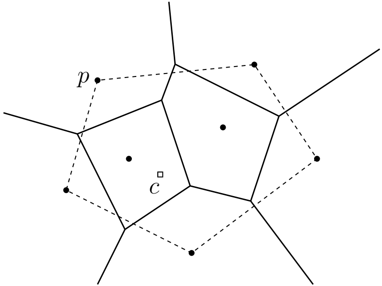

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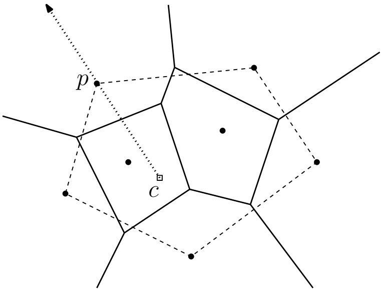

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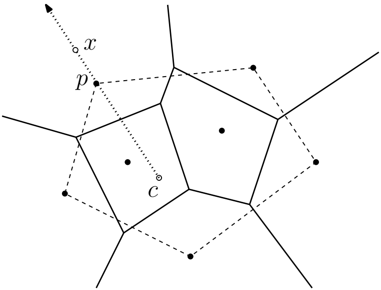
  - For any point $x \in \overrightarrow{cp} \setminus \overline{cp}$, x belongs to VR(p, S)



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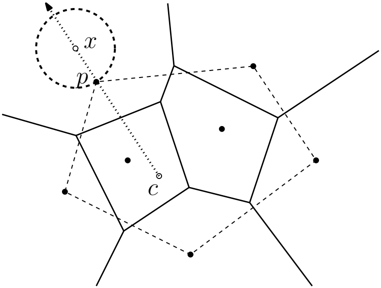

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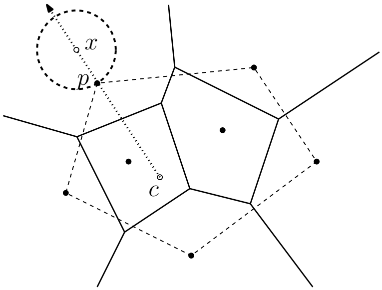
  - Select a point c in the convex hull
  - Shoot a ray $\overrightarrow{cp}$ from c to p
  - For any point $x \in \overrightarrow{cp} \setminus \overline{cp}$, x belongs to VR(p, S)
  - $\overrightarrow{cp}$ extends to the infinity.



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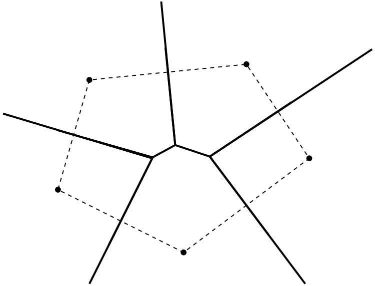
- If S is in convex position, $V(S)$ is a tree.



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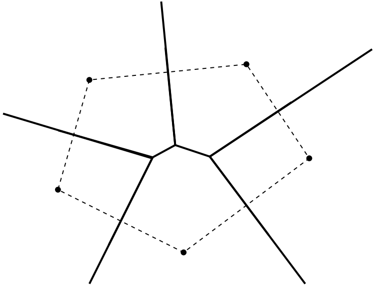
  - $\overrightarrow{cp}$ extends to the infinity.
- If S is in convex position, $V(S)$ is a tree.
- An unbounded Voronoi edge corresponds to a hull edge.



Voronoi Diagram (Mathematic Definition)

- Voronoi Diagram $V(S)$

$$V(S) = R^2 \setminus (\bigcup_{p \in S} \mathrm{VR}(p, S)) = \bigcup_{p \in S} \partial \mathrm{VR}(p, S)$$

  - $\partial \mathrm{VR}(p, S)$ is the boundary of $\mathrm{VR}(p, S)$
    - $\partial \mathrm{VR}(p, S) \not\subset \mathrm{VR}(p, S)$
  - $V(S)$ is the union of all the Voronoi edges
- Voronoi Edge e between $\mathrm{VR}(p, S)$ and $\mathrm{VR}(q, S)$

$$e = \partial \mathrm{VR}(p, S) \cap \partial \mathrm{VR}(q, S)$$

- Voronoi Vertex v among $\mathrm{VR}(p, S)$, $\mathrm{VR}(q, S)$, and $\mathrm{VR}(r, S)$

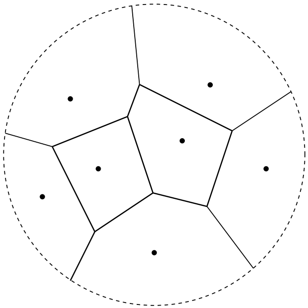
$$v = \partial \mathrm{VR}(p, S) \cap \partial \mathrm{VR}(q, S) \cap \partial \mathrm{VR}(r, S)$$

Complexity of $V(S)$

Theorem

$V(S)$ has $O(n)$ edges and vertices. The average number of edges of a Voronoi region is less than 6.

- Add a large curve Γ
  - Γ only passes through unbounded edges of $V(S)$
  - Cut unbounded pieces outside Γ
  - One additional face and several edges and vertices.



Complexity of $V(S)$

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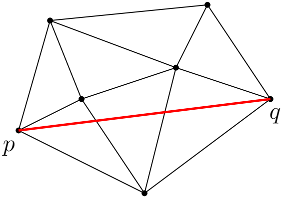
- Euler's Polyhedron Formula: $v - e + f = 1 + c$
  - v: # of vertices, e: # of edges, f: # of faces, and c: # number of connected components.
- An edge has two endpoints, and a vertex is incident to at least three edges.
  - $3v \leq 2e \rightarrow v \leq 2e/3$
- $f = n + 1$ and $c = 1$
  - $v = 1 + c + e - f = e + 1 - n \leq 2e/3 \rightarrow e \leq 3n - 3$
  - $e = v + f - 1 - c = v + n - 1 \geq 3v/2 \rightarrow v \leq 2n - 2$
- Average number of edges of a region $\leq (6n - 6)/n < 6$

Triangulation

Definition

Given a set S of points on the plane, a triangulation is maximal collection of non-crossing line segments among S.



Crossing ($\overline{pq}$)

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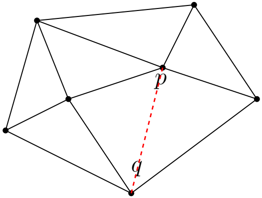

Not Maximal ($\overline{pq}$ is allowable)

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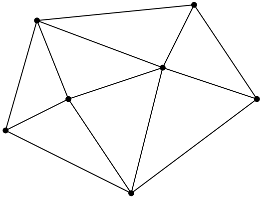
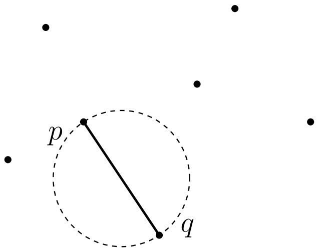


Triangulation

Delaunay Edge

Definition

An edge $\overline{pq}$ is called **Delaunay** if there exists a circle passing through p and q and containing no other point in its interior.



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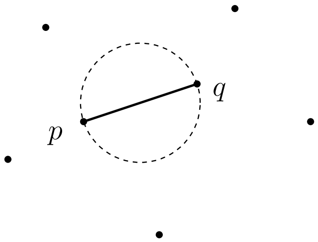

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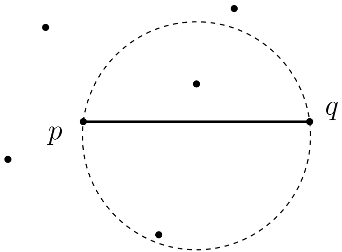
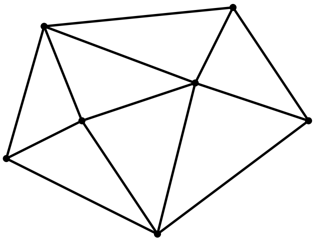


$\overline{pq}$ is **NOT** Delaunay

Definition

A **Delaunay Triangulation** is a triangulation whose edges are all Delaunay.



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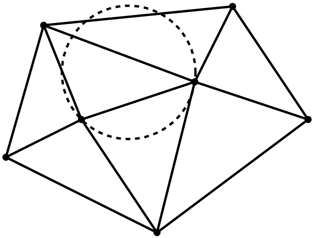

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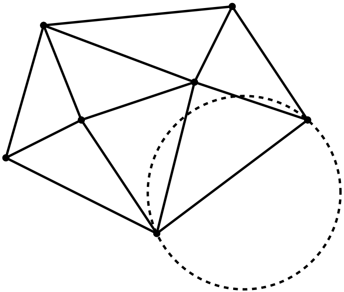

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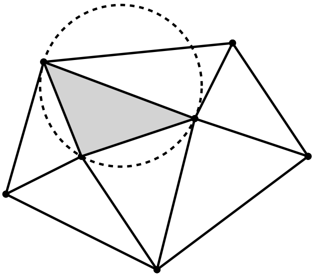
- For each face, there exists a circle passing all its vertices and containing no other point.



General Position Assumption

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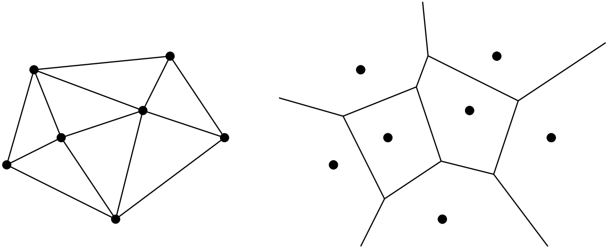
- There is a unique Delaunay triangulation.

Duality

Theorem

Under the general position assumption, the Delaunay triangulation is a dual graph of the Voronoi diagram.

- A site $p \leftrightarrow$ a Voronoi region $\mathrm{VR}(p, S)$



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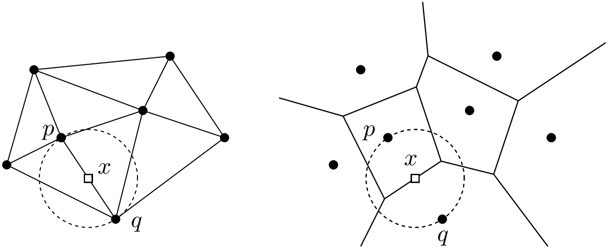

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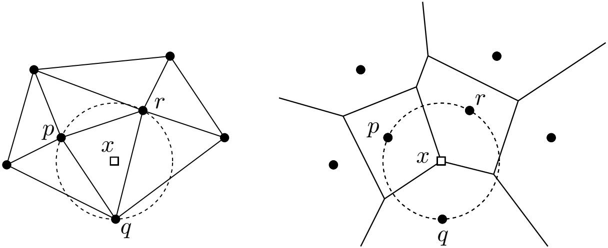
- A Delaunay triangle $\Delta pqr \leftrightarrow$ a Voronoi vertex among $VR(p, S)$, $VR(q, S)$ and $VR(r, S)$



Algorithms

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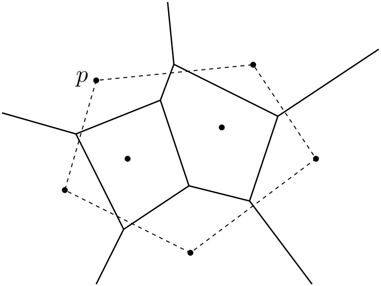

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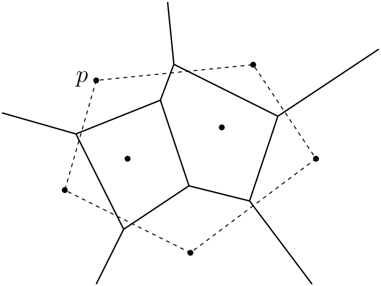


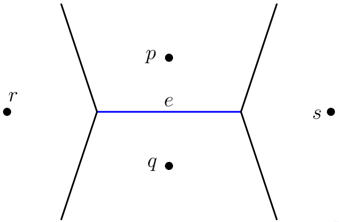
- $O(n \log n)$ time algorithms
  - Plane Sweep Algorithm
  - Divide and Conquer Algorithm

Randomized Incremental Construction

- General Idea
  - Consider a random sequence of S, $(s_1, s_2, \ldots, s_n)$.
  - Let R_i be $\{s_1, \ldots, s_i\}$
  - From $i = 4$ to $i = n - 1$, construct $V(R_{i+1})$ from $V(R_i)$ by inserting s_{i+1}.
- Tasks
  - What is a configuration?
  - What is a conflict relation?
  - How to use conflict relations to insert a site?
  - How to update conflict relations?
- General Position Assumption
  - No more than three sites are located on the same circle $\rightarrow$ The degree of a Voronoi vertex is exactly 3
  - No more than two points are located on the same line $\rightarrow$ The Voronoi diagram is connected

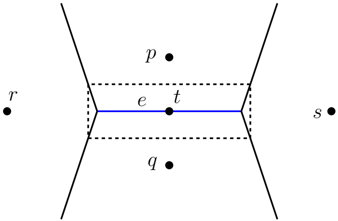
Configuration: A Voronoi edge

- A Voronoi region can not be a configuration because it could consist of $O(n)$ edges, i.e., it is not defined by a constant number of sites
- Consider a Voronoi edge e between $\mathrm{VR}(p, S)$ and $\mathrm{VR}(q, S)$
  - $e \subseteq B(p, q)$
  - Assume e has two endpoints v and u. Then $v = \overline{\mathrm{VR}(p, S)} \cap \overline{\mathrm{VR}(q, S)\mathrm{VR}(r, S)}$ and $u = \mathrm{VR}(p, S) \cap \mathrm{VR}(q, S)\mathrm{VR}(s, S)$.
  - e is defined by p, q, r, s
  - A Voronoi edge is defined by at most 4 sites.



Conflict Relation

- A site $t \in S \setminus R$ conflicts with a Voronoi edge e between $\mathrm{VR}(p, R)$ and $\mathrm{VR}(q, R)$ if $e \cap \mathrm{VR}(t, R \cup \{t\}) \neq \emptyset$.



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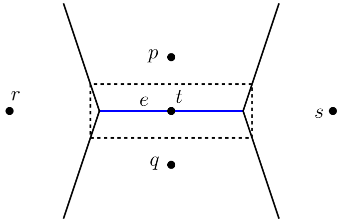
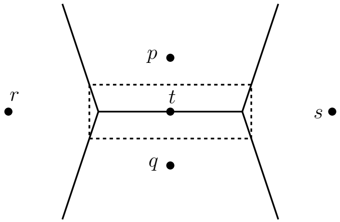


Lemma

$e \cap \mathrm{VR}(r, R \cup \{r\}) = e \cap \mathrm{VR}(r, \{p, q, r\})$ (Local Test)

Lemma

$V(R) \cap \mathrm{VR}(t, R \cup \{t\})$ is a tree



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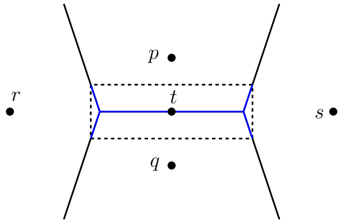

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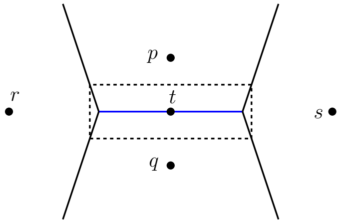

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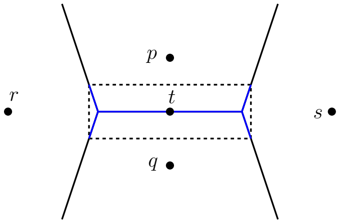

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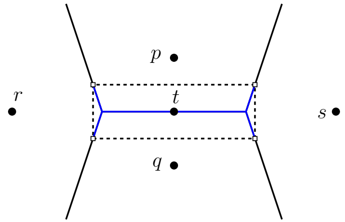

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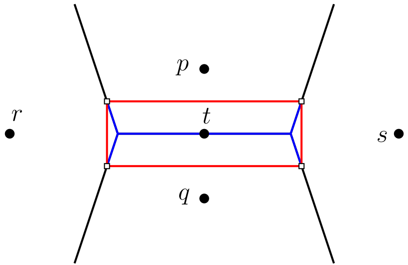

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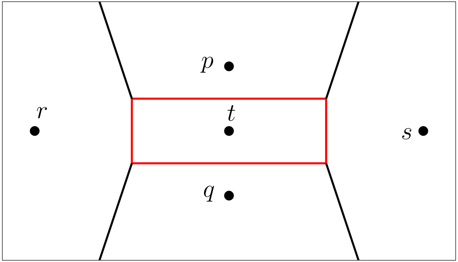

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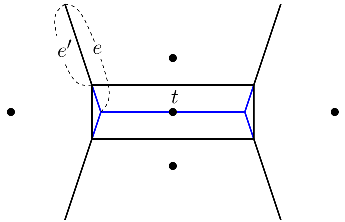
Update Conflict Relations: Partial Edges

- Consider an edge e' of $V(R \cup \{t\})$ which belongs to an edge e of $V(R)$



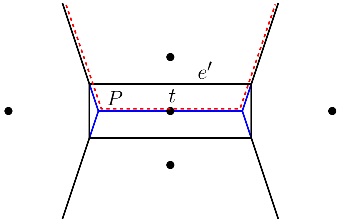
Lemma

Any site $s \in S \setminus (R \cup \{t\})$ in conflict with e' will conflict with e. That is, if $e' \cap \mathrm{VR}(t, R \cup \{t, s\} \neq \emptyset$, $e \cap \mathrm{VR}(s, R \cup \{t\}) \neq \emptyset$.

- The set of sites in conflict with e' is a subset of the set of sites in conflict with e
- For each site in conflict with e, check if it conflicts with e'.

Update Conflict Relations: Fully new edges

- Consider an edge e' of $V(R \cup \{t\})$ which does not belong to any edge of $V(R)$



Lemma

e' and a path of $V(R) \cap \mathrm{VR}(t, R \cup \{t\})$ will form a cycle. Let P be the path in $V(R) \cap \mathrm{VR}(t, R \cup \{t\})$ which forms a cycle with e'. Any site $s \in S \setminus (R \cup \{t\})$ in conflict with e' will conflict with one edge along the path.

- For each site in conflict with an edge of P, check if it conflicts with e'.

The number of updates

Lemma

Each edge of $V(R)$ which is destroyed due to the insertion of t will be check at most 3 times.

- An edge of $V(R)$ contains at most one edge $V(R \cup \{t\})$ and belongs to at most two paths which form a cycle with an edge of $V(R \cup \{t\})$.

Lemma

The time to insert t is proportional to the total size of the conflict lists for the edges of $V(R)$ which are destroyed due to the insertion of t

Thank You!!