Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Cop and Robber Game Cont./Randomizations

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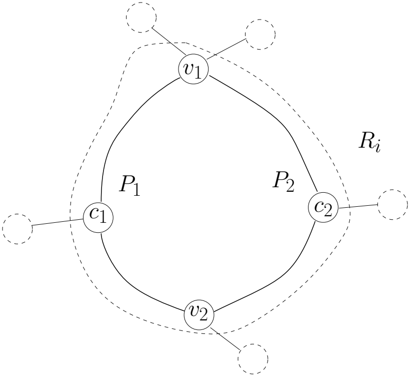
Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Proof:

- Two cops protect some paths, the third cop can proceed!



Number of cops required, positive result

Lemma 39: Consider a graph G and a shortest path $P = s, v_1, v_2, \ldots, v_n, t$ between two vertices s and t in G, assume that we have two cops. After a finite number of moves the path is protected by the cops so that after a visit of the robber R of a vertex of P the robber will be catched.

- Move cop c onto some vertex $c = v_i$ of P
- Assuming, r closer to some x in $s, v_1, \ldots, v_{i-1}$ *and* some y in $v_{i+1}, \ldots, v_n, t$. Contradiction shortest path from x and y
- $d(x, c) + d(y, c) \leq d(x, r) + d(r, y)$
- Move toward x, finally: $d(r, v) \geq d(c, v)$ for all $v \in P$
- Now robot moves, but we can repair all the time
- r goes to some vertex r' and we have $d(r', v) \geq d(r, v) - 1 \geq d(c, v) - 1$ for all $v \in P$.
- Some $v' \in P$ with $d(c, v') - 1 = d(r', v')$ exists, move to v'

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Proof:

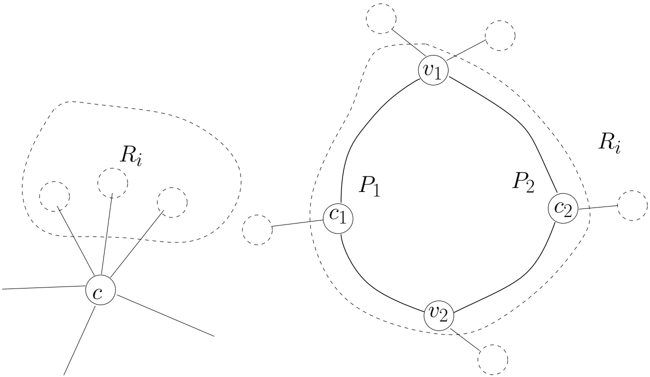
Case 1: All three cops occupy a single vertex c and the robber is located in one component R_i of $G \setminus \{c\}$

Case 2: There are two different paths P_1 and P_2 from v_1 to v_2 that are protected in the sense of Lemma 39 by cops c_1 and c_2. In this case $P_1 \cup P_2$ subdivided G into an interior, I, and an exterior region E. That is $G \setminus (P_1 \cup P_2)$ has at least two components. W.l.o.g. we assume that R is located in the exterior $E = R_i$.

Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 1 and Case 2



Theorem 40: For any planar graph G we have $c(G) \leq 3$.

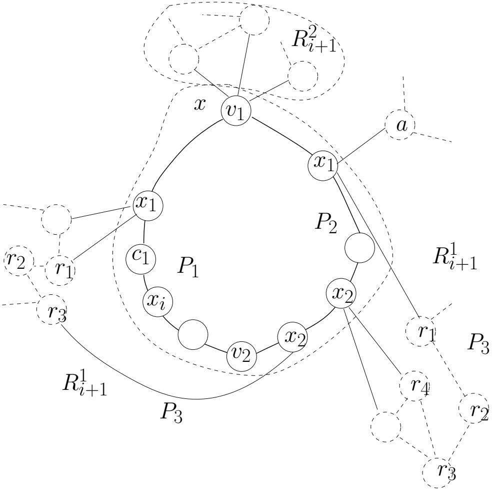
Case 1: Number of neighbors!

c one neighbor in R_i: Move all cops to this neighbor c' and Consider $R_{i+1} = R_i \setminus \{c'\}$. Case 1 again.

c more than one neighbor in R_i: a and b be two neighbors, $P(a, b)$ a shortest path in R_i between a and b. One cop remains in c, another cop protects the path $P(a, b)$ by Lemma 39. Thus $P_1 = a, c, b$ and $P_2 = P(a, b)$. Case 2 with $R_{i+1} \subset R_i$.

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 2:



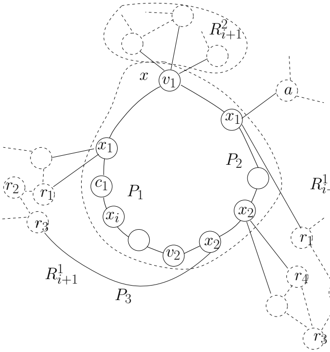
Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 2:

1. There is a another shortest path $P'(v_1, v_2)$ in $P_1 \cup P_2 \cup R_i$ but different from P_1 and P_2. Leaves $P_1 \cup P_2$ at x_1, hits $P_1 \cup P_2$ again at x_2.
2. There is no such path! There is a single vertex x of $P_1 \cup P_2$ so that R is in the component *behind* x. Move all three cops to x. Case 1 again!

Number of cops required, positive result

Shortest path $P'(v_1, v_2)$ in $P_1 \cup P_2 \cup R_i$ but different from P_1 and P_2. Leaves $P_1 \cup P_2$ at x_1, hits $P_1 \cup P_2$ again at x_2.



Let c_3 protect $P_3 = v_1, \ldots, x_1, r_1, \ldots, r_k, x_2, \ldots, v_2$ while c_1 and c_2 protect $P_1 \cup P_2$.

Case 2 again: c_3 protects P_3, c_1 or c_2 the remaining one!

Aspects of randomization

- Examples for the use of randomizations
- Context of decontaminations
- Randomization for a strategy
- Beat the greedy algorithm for trees
- Randomization as part of the variant
- Probability distribution for the root
- Expected number of vertices saved

Beat the greedy approximation

Integer LP formlation for trees (Exercise):

$$
\begin{array}{lrcll}
\text{Minimize} & \displaystyle\sum_{v\in V} x_v w_v & & & \\
\text{so that} & x_r = 0 & = & 0 & \\
 & \displaystyle\sum_{v \leq u} x_v & \leq & 1 & : \quad \text{for every leaf } u \\
 & \displaystyle\sum_{v\in L_i} x_v & \leq & 1 & : \quad \text{for every level } L_i, i \geq 1 \\
 & x_v & \in & \{0,1\} & : \quad \forall\, v \in V
\end{array}
$$

Strategy: Beat the greedy approximation

- $\mathtt{opt}_{ILP}$ optimal solution, $\mathtt{opt}_{RLP}$ fractional solution, $\mathtt{opt}_{ILP} \leq \mathtt{opt}_{RLP}$
- $\mathtt{opt}_{RLP}$ in polynomial time!
- Subtree T_v with $x_v = a \leq 1$ is *a-saved*, a portion $a \cdot w_v$ of the subtree is saved
- v_1 is ancestor of v_2 and $x_{v_1} = a_1$ and $x_{v_2} = a_2$
- Vertices of T_{v_2} are $(a_1 + a_2)$-saved. The remaining vertices of T_{v_1} are only a_1-saved.
- Randomized rounding scheme for every level
- Sum of the $x_v = a$-values for level i: Probability distribution for choosing v. Shuffle and set x_v to 1.
- Sum up to less than 1: Probability of not choosing a vertex at level i.
- Only problem: *double-protections*

Strategy: Beat the greedy approximation

- *double-protections*: Choose vertices on the same path to a leaf! We only use the predecessor! Skip the higher level!
- No such *double-protections*: The expected approximation value would be indeed 1.
- Intuitive idea: Tree T_{v_i} at level i is *fully* saved by the fractional strategy!
- Worst-case: Fractional strategy has assigned a $1/i$ fraction to all vertices on the path from r to v_i. This gives 1 for T_{v_i}.
- Probability of saving v_i is: $1 - (1 - 1/i)^i \geq 1 - \frac{1}{e}$.
- Formal general proof!

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\texttt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

- S_F fractional solution for $\texttt{opt}_{RLP}$
- Probabilistic rounding scheme: S_I outcome of this assignment
- Show: Expected protection of S_I is larger than $\left(1 - \frac{1}{e}\right)$ times the value of S_F
- x_v^F value of x_v for the fractional strategy
- x_v^I value $\{0, 1\}$ of integer strategy
- $y_v = \sum_{u \leq v} x_u \in \{0, 1\}$ indicate whether v is finally saved
- $y_v^F = \sum_{u \leq v} x_u^F \leq 1$ fraction of v saved by fractional strategy

Approximation by randomized strategy

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\mathtt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

For $y_v = 1$ it suffices that one of the predecessor of v was chosen. Let $r = v_0, v_1, v_2, \ldots, v_k = v$ be the path from r to v

$$\mathbf{Pr}[y_v = 1] = 1 - \prod_{i=1}^{k} (1 - x_{v_i}^F) \,.$$

Explanation: The probability that v_2 is safe is
$x_1 + (1 - x_1)x_2 = 1 - (1 - x_1)(1 - x_2)$
The probability that v_3 is safe is
$1-(1-x_1)(1-x_2)+(1-x_1)(1-x_2)x_3 = 1-(1-x_1)(1-x_2)(1-x_3)$
and so on.

Approximation by randomized strategy

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\mathtt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

$$
\begin{aligned}
\mathbf{Pr}[y_v = 1] &= 1 - \prod_{i=1}^{k}(1 - x_{v_i}^F) \\
&\geq 1 - \left(\frac{\sum_{i=1}^{k}(1 - x_{v_i}^F)}{k}\right)^k = 1 - \left(\frac{k - \sum_{i=1}^{k} x_{v_i}^F}{k}\right)^k \\
&= 1 - \left(\frac{k - y_v^F}{k}\right)^k \\
&= 1 - \left(1 - \frac{y_v^F}{k}\right)^k \geq 1 - e^{-y_v^F} \geq \left(1 - \frac{1}{e}\right) y_v^F .
\end{aligned}
$$

$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$

Approximation by randomized strategy

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\texttt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

$$\mathbf{E}(|S_I| = \sum_{v \in V} \mathbf{Pr}[y_v = 1] \geq \left(1 - \frac{1}{e}\right) \sum_{v \in V} y_v^F = \left(1 - \frac{1}{e}\right) |S_F| .$$

Randomization in variants of the problem

- $G = (V, E)$ fixed number k of agents
- k-surviving rate, $s_k(G)$, is the expectation of the *proportion* of vertices saved
- Any vertex is root vertex with the same probability
- Classes, C, of graphs G: For constant ϵ, $s_k(G) \geq \epsilon$
- Given G, k, $v \in V$ let:
  $\mathrm{sn}_k(G, v)$:number of vertices that can be protected by k agents, if the fire starts at v
- $\frac{1}{|V|} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \epsilon |V|$
- Class C: let the minimum number k that guarantees $s_k(G) > \epsilon$ for any $G \in C$ be denoted as the firefighter-number, $\mathrm{ffn}(C)$, of C.