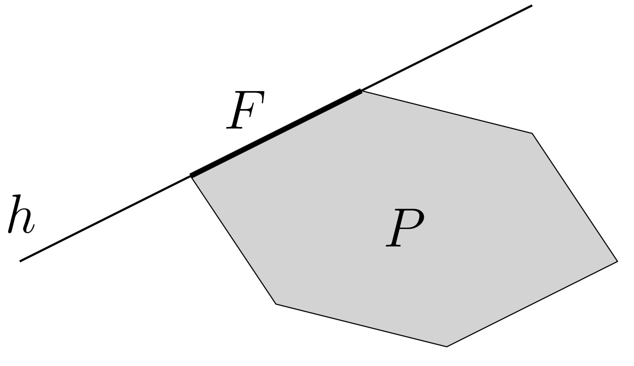
Faces of a Convex Polytope (Chapter 5.3)

Faces of the 3-dimensional cube

- 8 “corners” called vertices
- 12 edges
- 6 “squares” called facets

Definition (Face)
A face of a convex polytope P is defined as

- either P itself, or
- a subset of P of the form $P \cap h$, where h is a hyperplane such that P is fully contains in one of the closed half-spaces determined by h



Each face of a convex polytope P is a convex polytope

- P is the intersection of finitely many half-spaces, and h is the intersection of two half-spaces.
- So the face is an H-polyhedron, and it is bounded

A face of dimension j is called j-face. If P is a polytope of dimension d, then its faces have dimensions $-1, 0, 1, \ldots, d$, where -1 is the dimension of the empty set.

Names of faces

- 0-faces aree called *vertices*
- 1-faces are called *edges*
- $(d-1)$-faces are called *facets*
- $(d-2)$-faces are called *ridges*

The 3-dimensional cube has 28 faces in total: the empty face, 8 vertices, 12 edges (ridges), 6 facets, and the whole cube.

Definition of an *extremal point* of a set
For a set $X \subseteq \mathbb{R}^d$, a point $x \in X$ is *extremal* if $x \notin \operatorname{conv}(X \setminus \{x\})$

Main Proposition. Let $P \subset \mathbb{R}^d$ be a (bounded) convex polytope

(i) ("Vertices are extremal") The extremal points of P are exactly its vertices, and P is the convex hull of its vertices

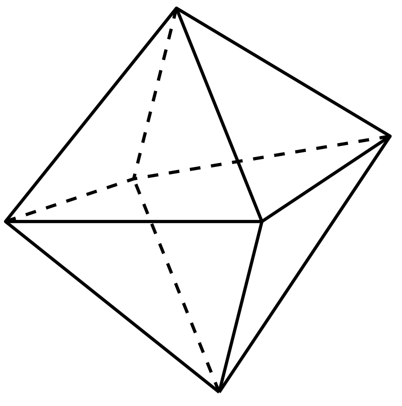
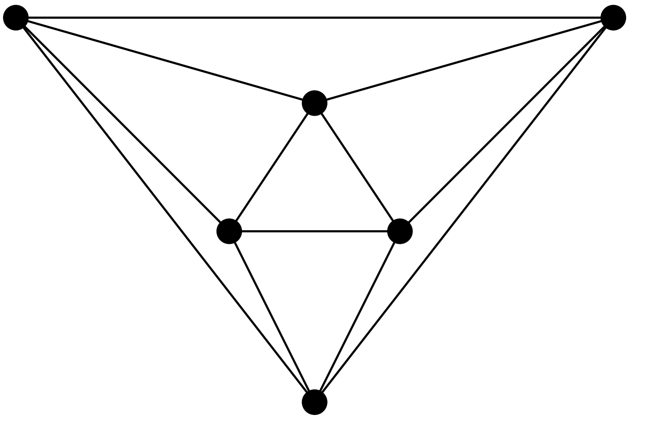
(ii) ("Face of a face is a face") Let F be a face of P. The vertices of F are exactly those vertices of P that lie in F. More generally, the faces of F are exactly those faces of P that are contained in F

The above proposition has two implications.

- Each V-polytope is the convex hull of its vertices
- The faces can be described combinatorially: they are convex hulls of certains subsets of vetices.

Graph of polytopes

- The vertices of the polytope are vetices of the graph
- Two vertices are connected by an edge in the graph if they are vertices of the same edge of P

For any convex polytope in $\mathbb{R}^3$, the graph is always planar

- Project the polytope from its interior point onto a circumscribed sphere
- make a "cartographic map" of this sphere, say stereographic projection

This graph is vertex 3-connected. (A graph G is called vertex k-connected if $|V(G)| \geq k+1$ and deleting any at most $k-1$ vertices leaves G connected.

Steinitz Theorem
A finite graph is isomorphic to the graph of a 3-dimensional convex polytope if and only if it is planar and vertex 3-connected.

Graphs of higher-dimensional polytopes probably have no nice description comparable to the 3-dimensional case.

- It is likely that the problem of deciding whether a given graph is isomorphic to a graph of a 4-dimensional convex polytope is NP-hard
- The graph of every d-dimenional polytope is vertex d connected (Balinski's theorem), but this is only a necessary condition

Examples

Simplex

A d-dimensional simplex has been defined as the convex hull of a $(d+1)$-point affinely independent set V.

- It is easy to see that each subset of V determines a face of the simplex
- There are $\binom{d+1}{k+1}$ faces of dimension k, $k = -1, 0, \ldots, d$, and 2^{d+1} faces in total

Crosspolytope

The d-dimensional crosspolytope has $V = \{e_1, -e_1, \ldots, e_d, -e_d\}$ as the vertex set.

- A proper subset $F \subset V$ determines a face if and only if there is no i such that both $e_i \in F$ and $-e_i \in F$
- There are 3^d+1 faces, including the empty one and the whole crosspolytope

Cube

The nonempty faces of the d-dimensional cube correspond to vectors $v \in \{-1, 1, 0\}^d$.

- The face corresponding to such v has the vertex set $\{u \in \{-1, 1\}^d \mid u_i = v_i$ for all i with $v_i \neq 0\}$.

Face Lattice

Let $\mathcal{F}(P)$ be the set of all faces of a (bounded) convex polytope P (including the empty face $\emptyset$ of dimension -1.

- We consider the partial ordering of $\mathcal{F}(P)$ by inclusion

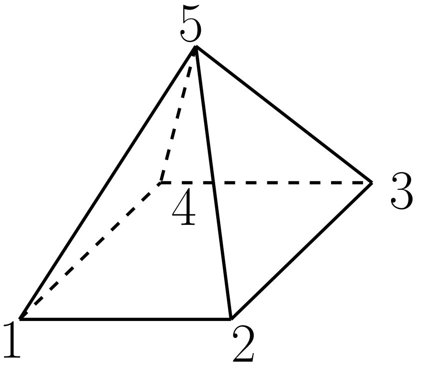
Definition of Combinatorial Equivalence

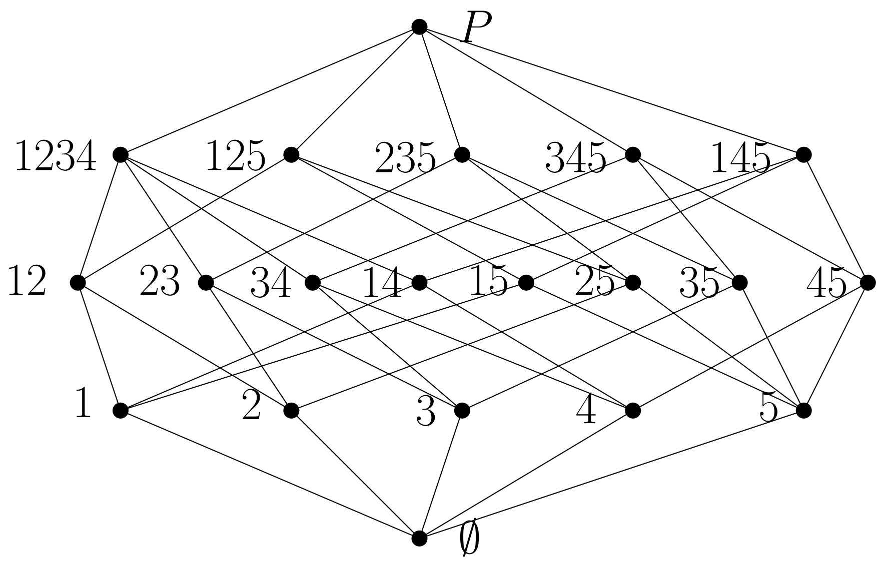
Two convex polytopes P and Q are called combinatorially equivalent if $\mathcal{F}(P)$ and $\mathcal{F}(Q)$ are isomorphic as partially ordered sets.

$\mathcal{F}(P)$ is a lattice (due to the partially ordered set). Recall the following two conditions:

- *Meet condition:* For any two faces $F, G \in \mathcal{F}(P)$, there exists a face $M \in \mathcal{F}(P)$, called the *meet* of F and G, that is contained in both F and G and contains all other faces contined in both F and G
- *Join condition:* For any two faces $F, G \in \mathcal{F}(P)$, there exists a face $J \in \mathcal{F}(P)$, called *join* of F and G, that contains both F and G and is contained in all other faces containing both F and G

The meet of two faces is their geometric intersection $F \cap G$.





The face lattice can be a suitable reprentation of a convex polytope in a computer

- Each j-face is connected by pointers to its (j-1)-faces and to the (j+1)-faces containing it
- It is a somewhat redundant representation
  - The vertex-facet incidences already contain the full information
  - For some applications, even less data may be sufficient, say the graph of the polytope

The dual polytope. Let P be a convex polytope containing the origin in its interior. Then the dual set P^* is also a polytope.

Propoistion
For each $j = -1, 0, \ldots, d$, the j-faces of P are in a bijective correspondence with the $(d-j-1)$-faces of P^*. This correspondence also reverses inclusion. The face lattices of P^* arises by turing the face of P upside down.

Example

- The cube and the octahedron are dual to each other
- the dodecahedron and the icosahedron are also dual to each other
- the tetrahedron is dual to itself.

If we have a 3-dimensional convex polytope and G is its graph, then the graph of the dual polytope is the dual graph to G.
More generall, we have

- The dual of a d-simplex is a d-simplex
- The d-dimensional cube and the d-dimensional crosspolytope are dual to each other.

Definition of Simple and Simplicial polytopes

- A polytope P is called *simplicial* if each of its facets is a simplex
  - This happens if the vertices of P are in general position, but general position is not necessary.
- A d-dimensional polytope P is called *simple* if each of its vertices are contained in exactly d facets.

Illustrations

- Since the faces of a simplex are again simplices, each proper face of a simplicial polytope is a simplex
- Simplicial polytopes: tetrahedron, octahedron, and icosahedron
- Simple polytopes: tetrahedron, cube, and dodecahedron
- 4-sided pyramid is neighbor simplicial nor simple

Duality

The dual of a simple polytope is simplicial, and vice versa

- For a simple d-dimensional polytope, s amll neighborhood of a vertex of the d-dimensional cube
- For each vertex v of a d-dimensional simple polytope, there are d edges enmanating from v, and each k-tuple of these edges uniquely determines one k-face incident to v.
- v belongs to $\binom{d}{k}$ k-faces, $k = 0, 1, \ldots, d$.