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slightly misses the goal while visiting ray i up to distance x_k. Instead, it finds the goal at step x_{J_k} on ray i arbitrarily close to $\beta_k x_k$. Either we have $x_{J_k} > \beta_k x_k$; that is, the searcher discovers the goal in distance x_{J_k} on ray i and moves $x_{J_k} - \beta_k x_k$ to the goal, or we have $x_{J_k} < \beta_k x_k$. In the latter case, the searcher moves $\beta_k x_k - x_{J_k}$ from x_{J_k} and finds the goal by accident. In both cases, the searcher moves $|x_{J_k} - \beta_k x_k|$ in the last step. Altogether, the competitive factor, $C(S)$, is bigger than

$$\frac{|x_{J_k} - \beta_k x_k| + \sum_{i=1}^{J_k-1} \beta_i x_i - x_i + \sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos\gamma_{i,i+1} + x_{i+1}^2}}{\beta_k x_k}.$$

By simple trigonometry, the shortest distance from $\beta_i x_i$ to a neighboring ray is given by $\beta_i x_i \sin\frac{2\pi}{n}$. Fortunately, this distance is smaller than the distance

$$\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos\gamma_{i,i+1} + x_{i+1}^2}$$

to any other ray. Thus, we have

$$C(S) > \frac{\sum_{i=1}^{J_k-1} \beta_i x_i}{\beta_k x_k} \sin\frac{2\pi}{n}.$$

Altogether, we have to find a lower bound for $\frac{\sum_{i=1}^{J_k-1} f_i}{f_k}$, where J_k denotes the index of the next visit of the ray of x_k and $f_i = \beta_i x_i$ denotes the search depth in step i. Fortunately, this problem is the same problem as in the competitive analysis for the usual m-ray problem where the searcher can move only along the rays. It was shown in Lemma 3.3 (see also Gal [Gal80] and Baeza-Yates et al. [BYCR93]) that for this problem there is an optimal strategy that visits the rays with increasing depth and in a periodic order; that is, $J_k = k+n$ and $i = k$. Applying Theorem 3.2 the best achievable strategy is given by $f_i = (n/(n-1))^i$. Altogether, this results in a function

$$(n-1)\left(\frac{n}{n-1}\right)^n \sin\frac{2\pi}{n}$$

for n rays. We can make n arbitrarily big because our construction is valid for every n. Note that we also have a lower bound for the problem of searching a point in the plane; this lower bound is close to the factor that is achieved by a spiral search.

Theorem 3.11 *For the ray search problem there is no strategy that achieves a better factor than*

$$\lim_{n\to\infty}(n-1)\left(\frac{n}{n-1}\right)^n \sin\frac{2\pi}{n} = 17.079\ldots$$

Additionally, every strategy for searching a point in the plane achieves a competitive factor bigger then 17.079… *(the optimal spiral achieves a factor of* 17.289… *[Gal80]).*

3.3 Searching in street polygons

Now we consider a special class of polygons, such that a competitive search still can be performed. By the m-ray search problem we already know that a constant competitive strategy for searching a point in arbitrary polygons does not exist.

The following polygons resembles streets or rivers where the path to the endpoint is not arbitrary although the path can make many windings and there are many caves where the goal might be loacated. Formally, we define a street polygon as follows:

Definition 3.12 Let P be a simple polygon with two points s and t on the boundary. P is denoted as a **street** (polygon), if the two boundary chains P_L and P_R of P between s and t are weakly visible, i.e., any point from P_L sees at least one point from P_R and vice versa. [Kle91]

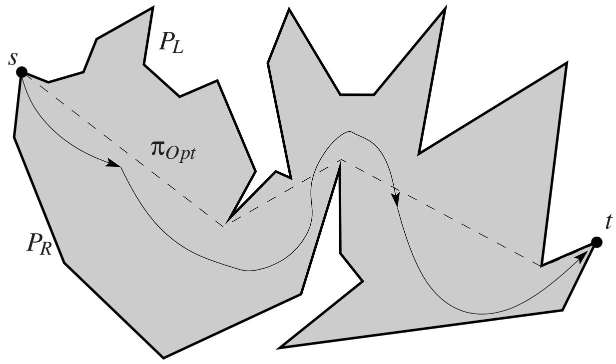

Figure 3.17: A street polygon.

Figure 3.17 shows an example. The main idea is that the shortest path from s to t also sees the boundary chains. Intuitively, if you use a street efficiently, you will always see the boundary chains.

Many structural properties have been proved for street polygons. For example, for a given polygon P one is interested on all possible pair of points (s,t) such that P is a street polygon. Surprisingly, this problem can be computed in linear time; see [THL98, DHN97]. In this section we consider the searching problem. That is, the start point s is given, the agent is equipped with a vision system and we are searching for a target t. The only information is, that P is a street for s and t. Against the shortest path to t we are searching for a competitive strategy with small ratio.

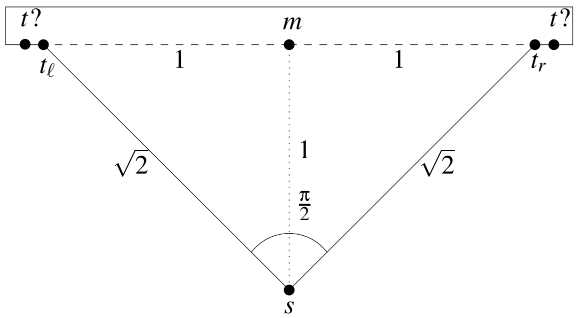

Figure 3.18: Lower bound for searching the target t.

A lower bound for the ratio in our problem can be constructed as follows.

Theorem 3.13 *(Klein, 1991)*
There is no strategy that finds the target t in a street with a path of length smaller than $\sqrt{2} \cdot \pi_{\text{Opt}}$. The competitive ratio is at least $\sqrt{2}$. *[Kle91]*

Proof. Consider Figure 3.18. The agent is located at s and*sees* t_ℓ and t_r. The target t lies behind one of them but the agent can only detect t if the line between t_ℓ and t_r is visited. Then the agent can move to t. If the agent visits the segmet between t_ℓ and t_r to the left (right) of the midpoint m, the target is positioned at the right (left). Thus the best the agent can achieve is moving directly to m. Thus we have (where $\varepsilon \longrightarrow 0$):

$$|\pi_{\text{Rob}}| = 2 \quad \text{und} \quad \frac{|\pi_{\text{Rob}}|}{|\pi_{\text{Opt}}|} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

□

In search of t we can make use of some structural properties. Consider Figure 3.19(i). The agent is located at s and does not see the caves (the shaded parts). A cave is generated by a corresponding reflex vertex[2] of the polygon. We can subdivide the current cave generating reflex vertices into the set of left

[2]Vertices, with inner angle $> \pi$.

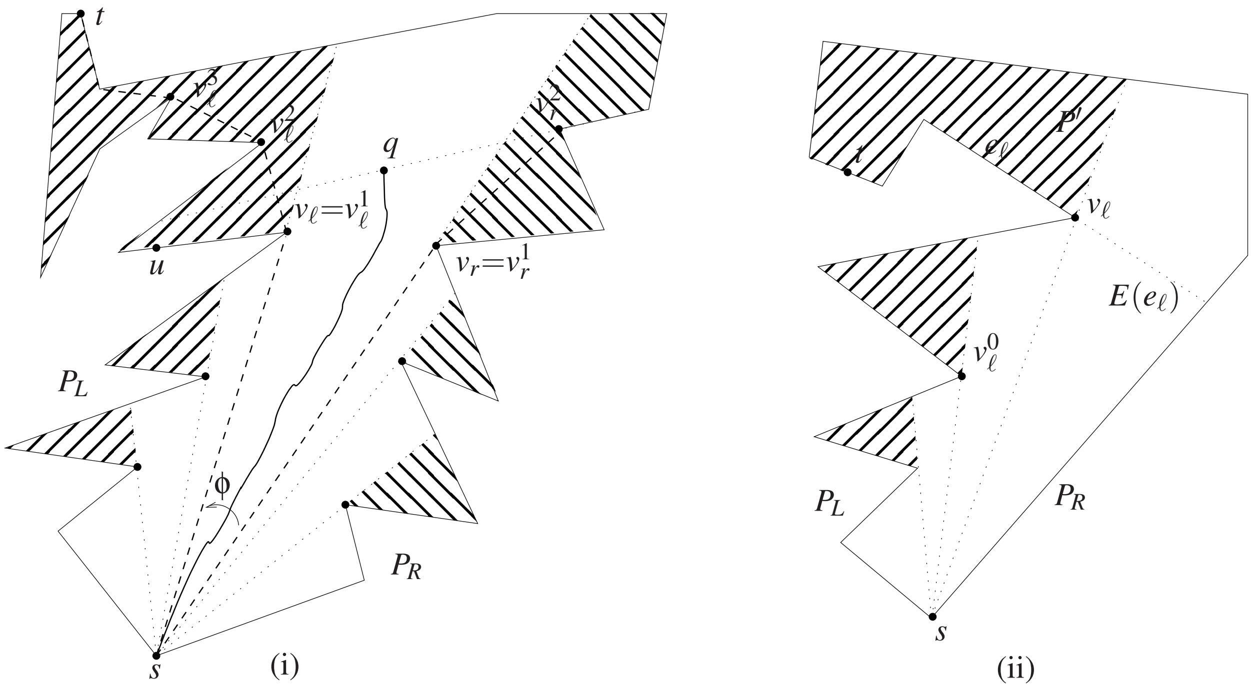

Figure 3.19: Typical situations for the task of searching the target in a street polygon.

reflex vertices (the cave is to the left) and right reflex vertices (the cave lies to the right). We call the vertices left or right reflex vertices, respectively.

Furthermore, we can consider the left reflex vertices in clockwise and the right reflex vertices in counter-clockwise order. One of these sequences can also be empty; in Figure 3.19(ii) there are no right reflex vertices.

We would like to argument that the unknown target t can only be located behind the rightmost left reflex vertex, say v_l, or the leftmost right reflex vertex, say v_r. The target cannot be located in one of the other caves. Assume that this is not the case. Assume that for example in Figure 3.19(i) the target is in the cave below v_l. In this case there is a point u on the right chain closely after v_l that does only see points on the right chain. This means that any reasonable strategy can concentrate on the current triangle of c, v_l and v_r, where c is the current location of the agent. It only makes sense to run into this triangle and let the opening angle at c increase.

If there is only one vertex v_l or v_r, it is clear that the target can only lie behind this remaining vertex and any reasonable strategy move directly to this vertex. It is also clear the the shortest path to the target has to run over this vertex. The same holds, when the target gets visible. The agent directly moves toward it.

Formally, we consider the following cases or events while the agent moves into the triangle of c, v_l and v_r.

- The target becomes visible. The agent moves toward it.
- The cave behing v_ℓ or v_r becomes visible and does not contain the target; as in point q in Figure 3.19(i). The goal has to be behind the remaining vertex, the agent directly moves toward it.
- Behind the current vertex v_ℓ or v_r another left or right reflex vertex becomes visible. For example v_ℓ^2 appears behind v_ℓ. In this case the current left reflex vertex changes from v_ℓ to v_ℓ^2. The agent runs into the triangle of c, v_ℓ^2 and v_r

The last event successively builds segments of convex chain constructed form reflex vertices $v_\ell^1, v_\ell^2, v_\ell^3, \ldots, v_\ell^i$ and $v_r^1, v_r^2, v_r^3, \ldots, v_r^j$ to the left and to the right starting from s. The agent only moves inside these two

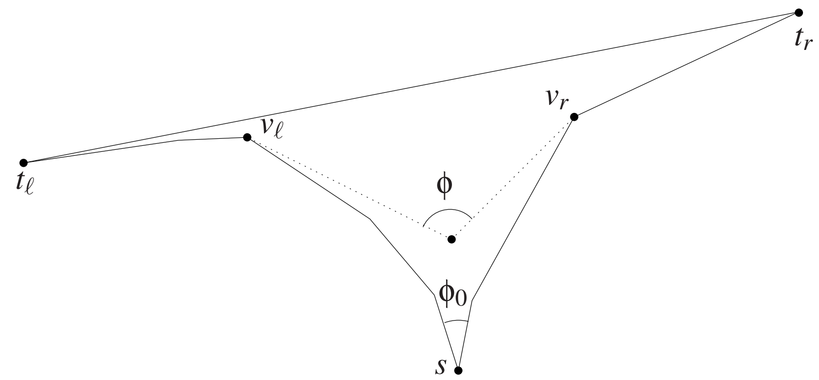

Figure 3.20: A funnel polygon.

chains. Therefore for simplicity we simply forget the original caves and only consider such funnel situations or so called funnel polygons. Beginning from s we have two convex chains that are finally closed by a segment t_l and t_r as shown in Figure 3.20. We assume that the current goal is either behind t_l or t_r. Actually there are two also caves behind t_l and t_r Altogether the funnel polygons will invoke the same path as in the original polygon with caves.

These funnel situations are the only situations that can provoke a detour. If one such situation is resolved, either the goal is reached or the agent is located at a point on the shortest path to the goal. This means that we can consider this situation as the main challenge. If we can guarantee a competitive ratio of C for any single funnel, we can combine the path to a C-competitive strategy in total.

Therefore we concentrate on such polygons.

Definition 3.14 A simple polygon is constructed by two convex chains P_L and P_R starting at a convex vertex s. The polygon can be closed by the segement $\overline{t_\ell t_r}$ of the endpoints of the chains; see seeFigure 3.20. such a polygon is denoted as a **funnel (polygon)**,

Another important observation for the exploration of the funnel is, that the opening angle ϕ for the current position c and the current active reflex vertices v_ℓ und v_r will increase monotonically for any reasonable strategy. The agent starts with a opening angle ϕ_0 at s and finally we will reach $\overline{t_\ell t_r}$ with opening angle 180°. Therefore it is quite natural to describe or parameterise a strategy by the opening angle ϕ.

First, we define a more general lower bound dedicated to the opening angle l ϕ. We can generalize Theorem 3.13 as follows:

Lemma 3.15 *For a funnel polygon with opening angle $\phi \leq \pi$ there is no strategy that has smaller path lenght than $K_\phi \cdot |\pi_{\mathrm{Opt}}|$ against the shortest path to the goal, where*

$$K_\phi := \sqrt{1+\sin\phi}.$$

Any strategy is at least K_ϕ competitive.

Proof. Consider Figure 3.21. By the sam argument as in the proof of Theorem 3.13 the best an agent can do is moving directly to the midpoint m. Any other movement results in a larger detour since we can place the target afterwards. Now the agent sees the target and moves toward it. [3]. For $\phi \leq \pi$ we have

$$\frac{|\pi_S|}{|\pi_{\mathrm{Opt}}|} = \frac{\ell\cos\frac{\phi}{2} + \ell\sin\frac{\phi}{2}}{\ell} = \sqrt{1+\sin\phi}.$$

□

[3]The path of length ε from v_ℓ orv_r to t need not be considered

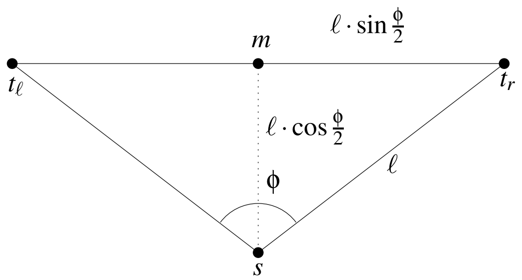

Figure 3.21: Generalized lower bound.

Note that for the final opening angle $\phi = \pi$ and $K_\phi = 1$ the agent will always move corretly, since the target is visible now. For $\phi = \frac{\pi}{2}$ we have the ratio $K_\phi = \sqrt{2}$ as in Theorem 3.13. For $0 \leq \phi \leq \pi$ the function K_ϕ gives a curve that starts at 1 rises up monotonically to $\sqrt{2}$ at $\frac{\pi}{2}$ and decreases monotonically toward 1 at π.

Assume that the agent explores a funnel starting from s with opening angle ϕ_0 and follows a path with monotonically increasing opening angles until $\overline{t_\ell t_r}$ is visited and $\phi = \pi$ holds.

For $\frac{\pi}{2} \leq \phi_1 < \phi_2$ we have $K_{\phi_1} > K_{\phi_2}$, and the competitive ratio for the overall exploration is dominated by the smaller angle. For $\phi_1 < \phi_2 \leq \frac{\pi}{2}$ we have $K_{\phi_1} < K_{\phi_2} \leq \sqrt{2}$, the ratio is dominated by the larger opening angle If the agent starts from an opening angle $\phi_0 < \frac{\pi}{2}$ along a path to angle $\phi = \pi$ there will always be a point such that the opnening angle $\phi = \frac{\pi}{2}$ is attained. Therefore the worst case ratio $\sqrt{2}$ is always included.

It seems to make sense to consider the case $\phi_0 < \frac{\pi}{2}$ and $\phi_0 \geq \frac{\pi}{2}$ separately. We start with $\phi_0 \geq \frac{\pi}{2}$. We already have a successful strategy for $\phi = \pi$. The following idea is that we apply a backward analysis that tells us how to prolong a successful strategy for opening angle ϕ_2 to a successful strategy for opening angle $\phi_1 < \phi_2$. By the following lemma we design a requirement for any path w from angle ϕ_1 to ϕ_2.

Lemma 3.16 *Let Π be a strategy that can reach the target of any funnel polygon with opening angle $\phi_2 \geq \frac{\pi}{2}$ by competitve ratio K_{ϕ_2}. We can extend this strategy to a K_{ϕ_1} competitive startegy for funnel polygons with opening angle ϕ_1 with $\phi_2 > \phi_1 \geq \frac{\pi}{2}$, if the path w between the two corresponding points fulfils the length condition Equation 3.9 for the current situation as depicted in Figure 3.22.*

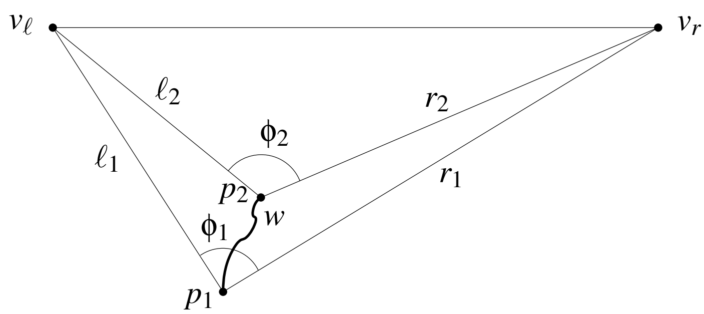

Figure 3.22: A path w from p_1 with angle ϕ_1 to p_2 with angle ϕ_2.

Proof. We consider a triangle with opening angle ϕ_1, start point p_1 and a path w to a point p_2 with opening angle ϕ_2; see Figure 3.22. From p_2 the agent can use the strategy Π for the angle ϕ_2 which is known by assumption. Π is K_{ϕ_2} competitive. Let us assume that during the movement w the vertices v_ℓ and v_r do not change.

Let ℓ_1 and ℓ_2 denote the distances from p_1 and p_2 to v_ℓ, as depicted in Figure 3.22, r_1 and r_2 are defined analogously. If the goal lies behind v_ℓ we can assume that the overall path length for $\pi^t_{p_1}$ from p_1 to t is:

$$|\pi^t_{p_1}| \leq |w| + K_{\phi_2} \cdot \ell_2.$$

We would like to guarantee that the overall strategy is K_{ϕ_1}-competitive, therefore we require: $K_{\phi_1} = \frac{|\pi'_{p_1}|}{|\pi_{\text{Opt}}|} \geq \frac{|w|+K_{\phi_2}\cdot\ell_2}{\ell_1}$, also

$$K_{\phi_1}\cdot\ell_1 \geq |w| + K_{\phi_2}\cdot\ell_2.$$

Analogously, if the goal is behind v_r, we require $K_{\phi_1}\cdot r_1 \geq |w| + K_{\phi_2}\cdot r_2$.

If we can guarantee that the path w from p_1 to p_2 fulfils the length condition

$$|w| \leq \min\{K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2\ ,\ K_{\phi_1}r_1 - K_{\phi_2}r_2\}, \tag{3.9}$$

we conclude that the overall strategy starting at p_1 attains a competitive ratio of K_{ϕ_1} for the funnel with opening angle ϕ_1.

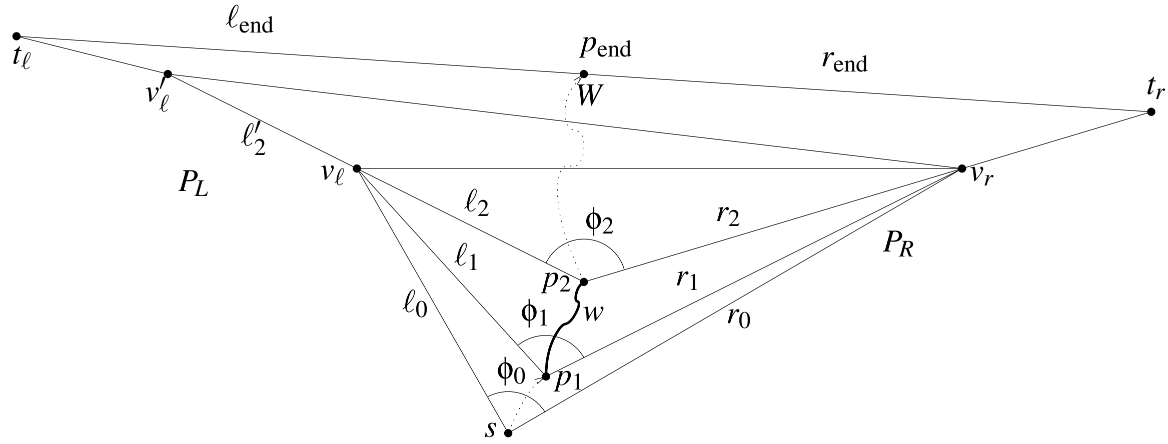

Figure 3.23: At p_2 a new left reflex vertex is detected.

Now it is clear that from time to time the reflex vertices in the funnel will change. The path w and the condition Equation 3.9 should still guarantee the above conclusion. Therefore we consider the situation that condition Equation 3.9 if fulfilled but precisely at p_2 there is a change of the reflex vertices as shown in Figure 3.23. In p_2 behind v_ℓ a new left reflex vertex v'_ℓ appears. Since Equation 3.9 holds we can conclude:

$$\begin{aligned} |w| &\leq K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2 \\ &= K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2 + K_{\phi_2}\ell'_2 - K_{\phi_2}\ell'_2 \\ &\leq K_{\phi_1}(\ell_1+\ell'_2) - K_{\phi_2}(\ell_2+\ell'_2) \end{aligned} \tag{3.10}$$

The last inequalityl is true, since from Lemma 3.15 for $\phi_2 > \phi_1 \geq \frac{\pi}{2}$ we have $K_{\phi_2} < K_{\phi_1}$. Note that $\ell_1+\ell'_2$ respectively $\ell_2+\ell'_2$ denote the lengths of the shortest paths from p_1 respectively p_2 to v'_ℓ. Equation Equation 3.10 says that the condition Equation 3.9 takes care that also for changes of the reflex vertices, we have obtain a K_{ϕ_1} competitive strategy at p_1 □

Assume that Equation 3.9 holds for all small changes of opening angles for the overall path W from s t op_{end}, we conclude

$$|W| \leq \min\{K_{\phi_0}\cdot|P_L| - K_\pi\ell_{\text{End}}\ ,\ K_{\phi_0}\cdot|P_R| - K_\pi r_{\text{End}}\}.$$

Altogether we have a K_{ϕ_0} competitive strategy in this case.

Now it is sufficient to guarantee that the agent fulfils Equation 3.9 during the movements. The idea of fulfilling this requirement is as follows: The portions $K_{\phi_1}\ell_1 - K_{\phi_2}\ell_2$ and $K_{\phi_1}r_1 - K_{\phi_2}r_2$ somehow express how many path length w we can use in the next step for the left or the right location of the goal, respectively. Since we do not know where the target will be at the end, we do not want to let one side have an advantage at this stage.

Therefore we would like to guarantee that both values are the same. This gives

$$K_{\phi_2}(\ell_2 - r_2) = K_{\phi_1}(\ell_1 - r_1)\,.$$

Fortunately, by this requirement we indeed define a special curve for any starting situation with angle ϕ_0 and length l_0 and r_0. Let $A = K_{\phi_0}(\ell_0 - r_0)$ The curve that fulfils the above equation all the time is given by

$$\begin{aligned} X(\phi) &= \frac{A}{2} \cdot \frac{\cot\frac{\phi}{2}}{1+\sin\phi} \cdot \sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2} \\ Y(\phi) &= \frac{1}{2} \cdot \cot\frac{\phi}{2} \cdot \left(\frac{A^2}{1+\sin\phi} - 1\right). \end{aligned}$$

We will now explain how we have developed the formulas above.We choose a coordinate system with axis paralell to $v_l\,v_r$, the midpoint of $v_l\,v_r$ is the origin. We scale such that $|v_l\,v_r| = 1$. Let p be the point on the curve with opening angle ϕ; see Figure 3.24. We have starting values ϕ_0, l_0 and r_0 and set $A := K_{\phi_0}(\ell_0 - r_0)$.

In order to find p we have to fulfil two conditions. First, the difference $l(p) - r(p)$ of the distances from p to v_l and v_r has to equal $\frac{A}{K_\phi}$. The locus of all such point is a hyoerbola. Second the angle at p with respect to v_l and v_r has to be ϕ. The locus of all such points is a circle; see Figure 3.24. This holds because of the Thales' circle property.

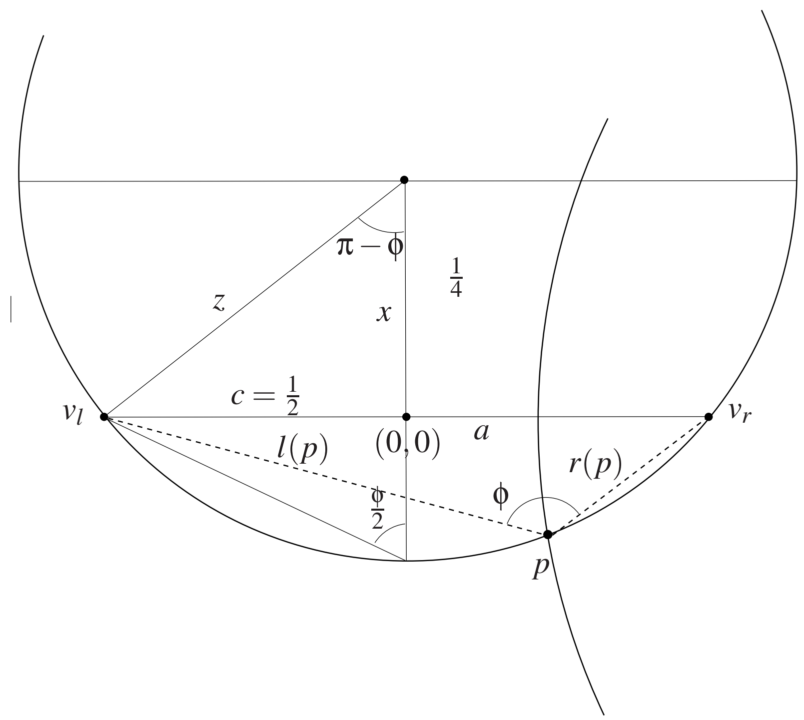

Figure 3.24: The left arc of the hyperbola is defined by v_l, v_r and $(l(p) - r(p)) = \frac{A}{K_\phi}$ and the circle running through v_l and v_r is defined by the opening angle ϕ.

The hyperbola is defined by

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1\,,$$

where $2a = (l(p) - r(p)) = \frac{A}{K_\phi}$ and $b^2 + a^2 = c^2 = \frac{1}{4}$ holds. This gives $a^2 = \left(\frac{A}{2K_\phi}\right)^2$ and $b^2 = \frac{1}{4} - \left(\frac{A}{2K_\phi}\right)^2$. The circle is defined by

$$X^2 + (Y - x)^2 = z^2 \,. \tag{3.10}$$

This means that we have to calculate x and z. From the law of sine we conclude

$$\begin{aligned}\frac{z}{\sin\frac{\pi}{2}} &= \frac{1}{2\sin(\pi - \phi)} = \frac{1}{2\sin\phi} \\ \frac{z - x}{\sin\left(\pi - \frac{\pi}{2} - \frac{\phi}{2}\right)} &= \frac{z - x}{\cos\frac{\phi}{2}} = \frac{1}{2\sin\frac{\phi}{2}}\end{aligned}$$

and therefore $z = \frac{1}{2\sin\phi}$ and

$$x = z - \frac{1}{2}\cot\frac{\phi}{2} = \frac{1}{2\sin\phi} - \frac{1}{2}\cot\frac{\phi}{2} = \frac{1 - 2\cos^2\frac{\phi}{2}}{4\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = -\frac{\cot\phi}{2} \,.$$

The intersection of the hyperbola and the circle is indeed given by the above functions $X(\phi)$ and $Y(\phi)$. We have found the solutions by a computer algebra system. Here we simply verify that the solutions are correct. We insert the values into the hyberboly and the circle description.

$$\frac{X^2}{\left(\frac{A}{2K_\phi}\right)^2} - \frac{Y^2}{\left(\frac{1}{2}\right)^2 - \left(\frac{A}{2K_\phi}\right)^2} = 1 \tag{3.11}$$

$$X^2 + \left(Y + \frac{\cot\phi}{2}\right)^2 = \frac{1}{4\sin^2\phi}. \tag{3.12}$$

For (3.11) we have

$$\begin{aligned}\frac{\left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\sqrt{\left(1+\tan\frac{\phi}{2}\right)^2 - A^2}\right)^2}{\left(\frac{A}{2K_\phi}\right)^2} - \frac{\left(\frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi} - 1\right)\right)^2}{\left(\frac{1}{2}\right)^2 - \left(\frac{A}{2K_\phi}\right)^2} &= \\ \left(\frac{\cot\frac{\phi}{2}}{K_\phi}\right)^2\left(\left(1+\tan\frac{\phi}{2}\right)^2 - A^2\right) - \frac{\cot^2\frac{\phi}{2}\left(\left(\frac{A}{K_\phi}\right)^2 - 1\right)^2}{1 - \left(\frac{A}{K_\phi}\right)^2} &= \\ \left(\frac{\cot\frac{\phi}{2}}{K_\phi}\right)^2\left(\left(1+\tan\frac{\phi}{2}\right)^2 - A^2\right) + \cot^2\frac{\phi}{2}\left(\left(\frac{A}{K_\phi}\right)^2 - 1\right) &= \\ \cot^2\frac{\phi}{2}\left(\frac{\left(1+\tan\frac{\phi}{2}\right)^2}{1+\sin\phi} - 1\right) &= 1\,.\end{aligned}$$

The conclusion is valid since the following identity holds.

$$1 + \sin\phi = 1 + \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{\left(1+\tan\frac{\phi}{2}\right)^2}{1+\tan^2\frac{\phi}{2}} \tag{3.13}$$

For showing (3.12) we proceed as follows:

$$\begin{aligned}
\left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\sqrt{\left(1+\tan\frac{\phi}{2}\right)^2-A^2}\right)^2+\left(\frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)+\frac{\cot\phi}{2}\right)^2 &= \\
\left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\right)^2\left(\left(1+\tan\frac{\phi}{2}\right)^2-A^2\right)+ & \\
\left(\frac{1}{2}\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)\right)^2+\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)\frac{\cot\phi}{2}+\left(\frac{\cot\phi}{2}\right)^2 &= \\
\left(\frac{A}{2}\cdot\frac{\cot\frac{\phi}{2}}{1+\sin\phi}\right)^2\left(1+\tan\frac{\phi}{2}\right)^2+ & \\
\left(\frac{1}{2}\cot\frac{\phi}{2}\right)^2\left(-2\frac{A^2}{1+\sin\phi}+1\right)+\cot\frac{\phi}{2}\left(\frac{A^2}{1+\sin\phi}-1\right)\frac{\cot\phi}{2}+\left(\frac{\cot\phi}{2}\right)^2 &= \\
\left(\frac{\cot\frac{\phi}{2}}{2}-\frac{\cot\phi}{2}\right)^2+\frac{A^2\cot^2\frac{\phi}{2}}{4(1+\sin\phi)}\left(\frac{\left(1+\tan\frac{\phi}{2}\right)^2}{1+\sin\phi}-2+2\frac{\cot\phi}{\cot\frac{\phi}{2}}\right) &= \\
\frac{1}{4\sin^2\phi}+\frac{A^2\cot^2\frac{\phi}{2}}{4(1+\sin\phi)}\left(\tan^2\frac{\phi}{2}+1-2+\frac{1-\tan^2\frac{\phi}{2}}{\tan\frac{\phi}{2}}\tan\frac{\phi}{2}\right) &= \\
\frac{1}{4\sin^2\phi}+\frac{A^2\cot^2\frac{\phi}{2}}{4(1+\sin\phi)}\cdot 0 &= \frac{1}{4\sin^2\phi}.
\end{aligned}$$

Here we make use of the identity (3.13) and the equations

$$\left(\frac{\cot\frac{\phi}{2}}{2}-\frac{\cot\phi}{2}\right)^2=\frac{1}{4}\left(\frac{\sin\phi}{1-\cos\phi}-\frac{\cos\phi}{\sin\phi}\right)^2=\frac{1}{4}\frac{1}{\sin^2\phi}$$

and

$$\cot\phi=\frac{1-\tan^2\frac{\phi}{2}}{2\tan\frac{\phi}{2}}.$$

Finally, we have to prove that the above curve indeed fulfils the condition for any small piece w. Experimentally, we make use of the precise curve description and import it into Geogebra or Maple. Here we approximate the path between any two points by the corresponding segment. This procedure already indicates that assumption has to be true.

It can also be shown analytically. A lengthy, detailed proof is given in [IKL99] or [Lan00]. Figure 3.25 shows examples for the curve for different values of ϕ and A. The figure stems from a Maple plot.

We obtain the following result:

Corollary 3.17 *For a funnel polygon with opening angle $\phi_0 > \frac{\pi}{2}$ we will find any unknown target within a competitive ratio K_{ϕ_0}.*

Finally, for angles $0 < \phi_0 < \frac{\pi}{2}$ we can apply the same approach. Of course we can also apply the condition

$$K_{\phi_2}(\ell_2 - r_2) = K_{\phi_1}(\ell_1 - r_1)$$

for $\phi_1 < \phi_2 < \frac{\pi}{2}$.

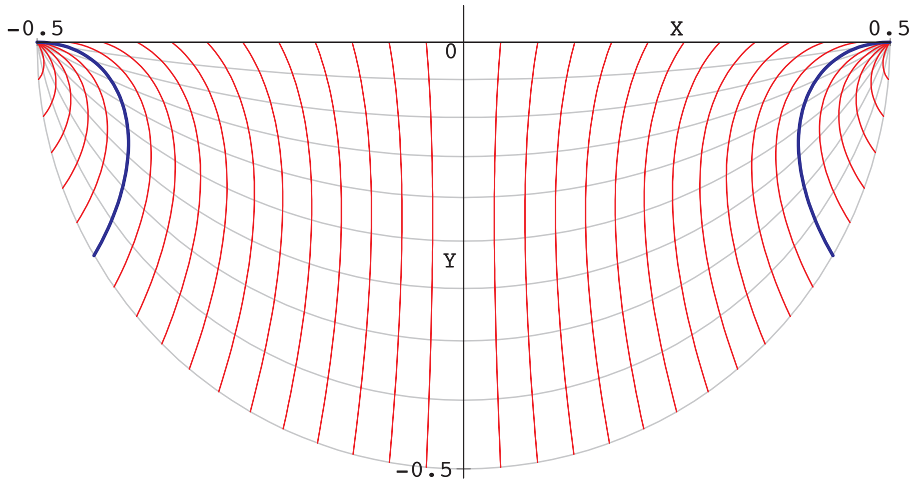

Figure 3.25: Curves $(X(\phi), Y(\phi))$ depending from ϕ and A.

Not that this will also result in a continuous extension of the curves of Figure 3.25. The problem is that these curve parts will not fulfil the condition Equation 3.9 because $K_{\phi_1} < K_{\phi_2}$ holds. Therefore we just insert the fixed ratio $\sqrt{2}$ which we would like to achieve at angle $\frac{\pi}{2}$. The factor $\sqrt{2}$ dominates all K_ϕ.

By the same arguments as before it is sufficient to guarantee

$$w \leq \min\{\sqrt{2}(\ell_1 - \ell_2)\ ,\ \sqrt{2}(r_1 - r_2)\}$$

for any small piece of our curve.

Again we would not prefer one side and set $\ell_1 - \ell_2 = r_1 - r_2$. This means that we are moving on the current angular bisector and call this startegy CAB (Current Angular Bisector); see also [IKL97, LOS96]. The analysis is also prenseted in [IKL99] oder [Lan00]. Note that if we apply the factor $\sqrt{2}$ for the angles above $\frac{\pi}{2}$ for the path w we will also define a curve but the above path length property for w does not hold.

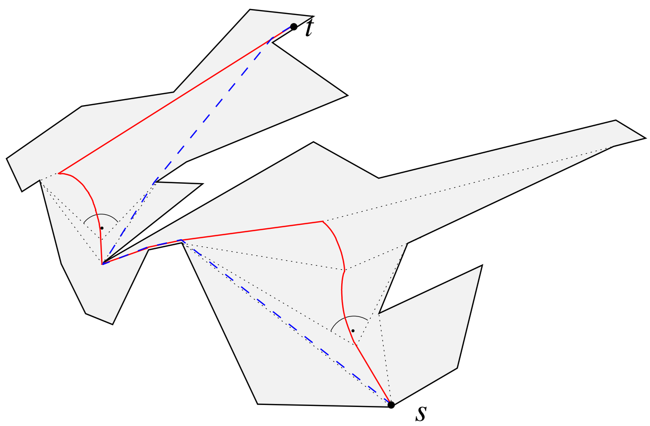

Figure 3.26: An example of the application of WCA.

Algorithm 3.1 summarizes the strategy, Figure 3.26 shows an example of its application. Altogether, the following result holds:

Theorem 3.18 *(Icking, Klein, Langetepe, Schuierer, Semrau, 1999)*
Searching for the target t inside an unknown street polygon can be performed by an optimal $\sqrt{2}$ competitive strategy. *[IKL99, SS99, IKL$^+$04]*

We have implemented the optimal strategy under the name “WCA” (Worst-Case-Aware), an applet can be found here:

`http://www.geometrylab.de/`

Algorithm 3.1 Searching for the target of a street.

While target t is not visible:

- Compute extreme reflex vertices v_ℓ and v_r.
- FIf only on exist, move toward it.
- Otherwise repeat:
  - If a new reflex vertex v'_ℓ or v'_r is detected: Replace v_ℓ or v_r by v'_ℓ or v'_r, respectively.
  - Let ϕ be the current opening angle w.r.t. v_ℓ and v_r.
  - If $\phi \leq \frac{\pi}{2}$: Follow the current angular bisector
  - If $\phi > \frac{\pi}{2}$: Follow the curve represented by $X(\phi)$ and $Y(\phi)$ with the current value A.
- Until either v_ℓ or v_r is fully explored. Move to the vertex on the opposite side.

Move to the target t.

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