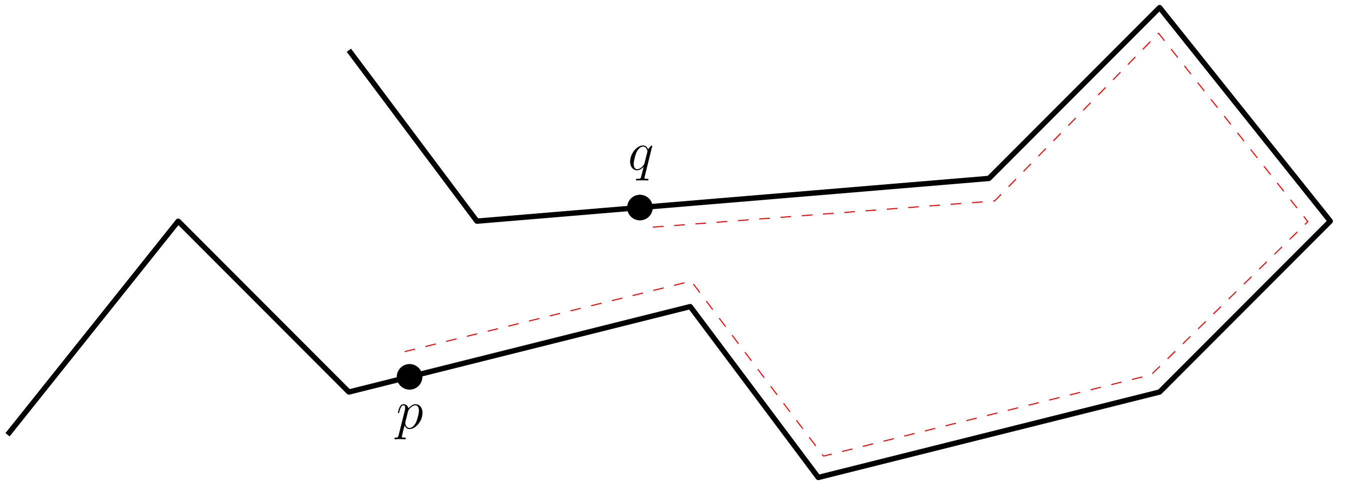
Geometric Dilation



A polygonal chain C

detour of C on the pair (p, q):

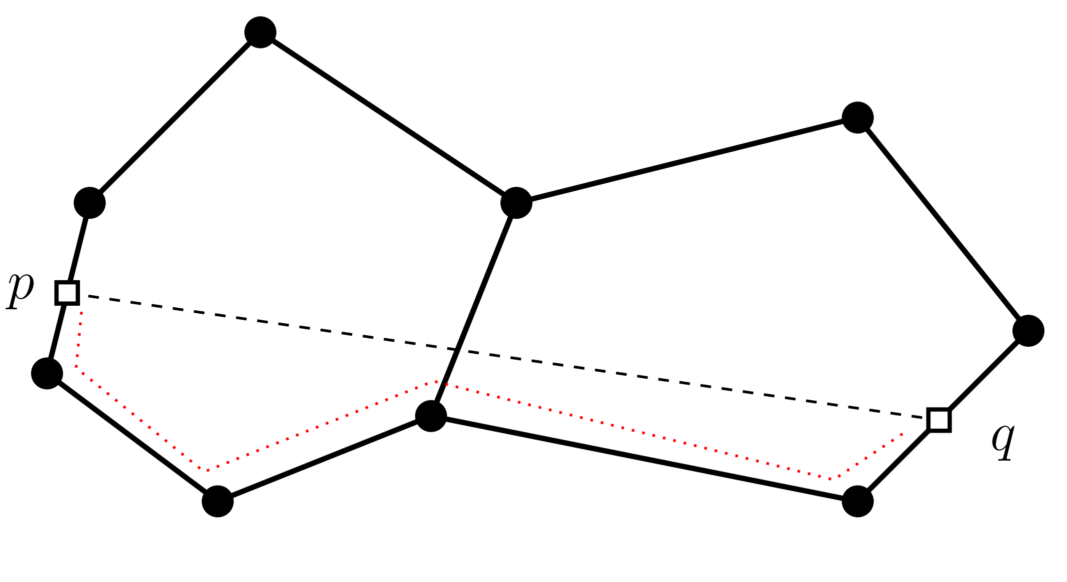
$$\delta(p, q) = \frac{|C_p^q|}{|pq|},$$

where C_p^q the path on C connecting p and q.

detour of C

$$\delta_C = \max_{p,q \in C} \delta(p, q).$$

More general: A connected planar graph $G(V, E)$



Gemetric Dilation of $G(V, E)$

$$\delta_G = \sup_{p \neq q} \frac{|\Pi_p^q|}{|pq|},$$

where Π_p^q is f the shortest path between p and q in G

worst detour of the network

Graph Dilation of $G(V, E)$

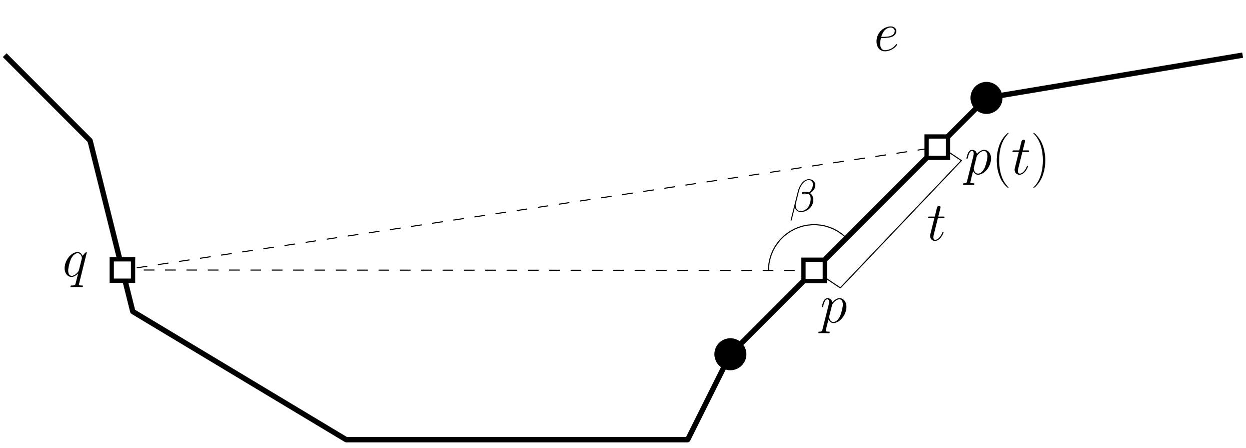
$$\sigma_G = \sup_{p \neq p, p, q \in V} \frac{|\Pi_p^q|}{|pq|},$$

where Π_p^q is the shortest path between p and q in G

worst detour among vertices

In this lecture, we will focus on the detour on a polygonal chain C

Try to find structural properties for the worst-case pair (p, q)



Fix a point $q \in C$, an edge e of C, and a point $p \in e$, and let $p(t)$ be a point in e that lies in distance $|t|$ from $p = p(0)$ in positive direction.

Where is the maximum detour for a fixed point q and a point in a fixed edge e?

Lemma 1

- moving p toward $p(t)$ decreases $\delta(p,q) \leftrightarrow \cos\beta < \frac{-|pq|}{|C_p^q|}$
- a local maximum at $p \leftrightarrow \cos\beta = \frac{-|pq|}{|C_p^q|}$
- moving p toward $p(t)$ increases $\delta(p,q) \leftrightarrow \cos\beta > \frac{-|pq|}{|C_p^q|}$

Proof
By the cosine law, we have

$$\delta(p(t),q) = \frac{t + |C_p^q|}{\sqrt{t^2 + |pq|^2 - 2t|pq|\cos\beta}}$$

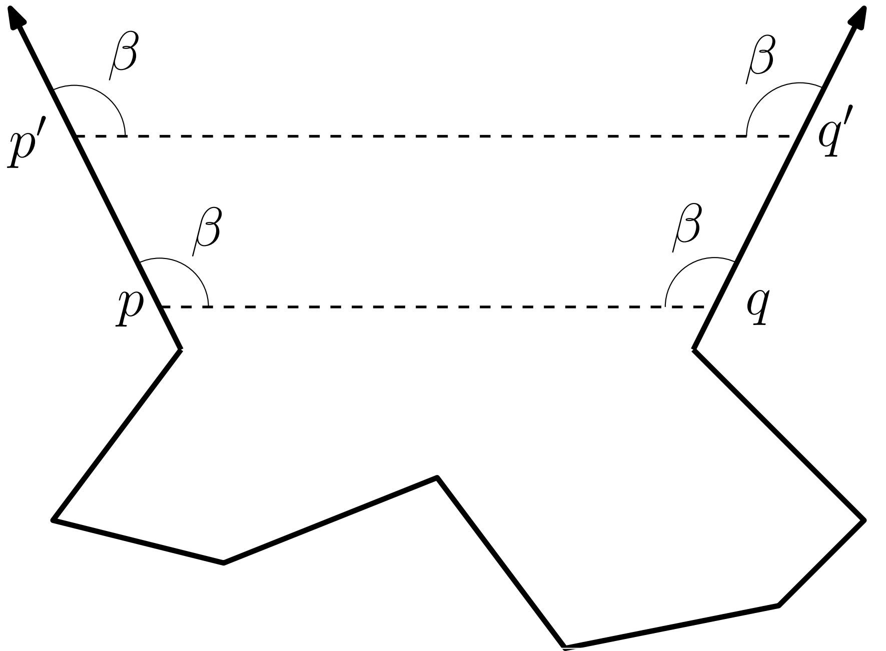
The derivative with respect to t: $\delta'(p(t),q) =$

$$\frac{\sqrt{t^2 + |pq|^2 - 2t|pq|\cos\beta} - (t + |C_p^q|)\frac{1}{2}\frac{1}{\sqrt{t^2+|pq|^2-2t|pq|\cos\beta}}(2t - 2|pq|\cos\beta)}{\sqrt{t^2 + |pq|^2 - 2t|pq|\cos\beta}^2}$$

When t is zero,

$$\frac{|pq|^2 - |C_p^q|(-|pq|\cos\beta)}{|pq|^2} = 1 + \frac{|C_p^q|}{|pq|}\cos\beta$$

Lemma 2
Any polygonal chain makes its maximum detour on a pair of points at least one of which is a vertex



By Lemma 1, the line segment pq must form the same angle,

$$\beta = \arccos(-\frac{|pq|}{|C_p^q|}),$$

with the two edges containing p and q. (Otherwise, mvoing one of the points can increase the detour).
Therefore, we can move both points simutaneously until one of them reaches the endpoints of its edges. In fact, we have

$$\delta(p', q') = \frac{|C_p^q| + 2t}{|pq| - 2t\cos\beta} = \frac{|C_p^q|}{|pq|} = \delta(p, q).$$

Direct Consequnece:
$\delta(C)$ can be computed in $O(n^2)$ time

- Let p_1, p_2, ..., p_n be the consecutive vertices of C.
- In $O(n)$ time, we can compute $|C_{p_1}^{p_i}|$ to every vertex p_i.
- For any two vertices p and q, $|C_p^q| = ||C_{p_1}^p| - |C_{p_1}^q||$ can be computed in $O(1)$ time
- For a vertex q of C and an edge e of C, we can compute the maximum detour between q and a point $p \in e$ in $O(1)$ time

Definition
Two points, p and q on C, are called *co-visible* if the line segment connecting them contains no points of the chain C in its interior.

Definition
For two co-visible points, p and q, if p is a vertex and q is an interior point of an edge or q is a vertex and p is an interior point of an edge, (p, q) is called a vertex-edge cut.

Lemma 3. The maximum detour of C is attained by a vertex-edge cut (p, q)

Proof

1. p and q are co-visible
   - Let $p = p_0, p_1, \ldots, p_k = q$ be the points of C intersected by the line segment pq, ordered by their appearance on pq.
   - For each pair (p_i, p_{i+1}) of consecutive points, let C_i denote the segment of C that connects them.
   - Since these segments need not be disjoint, $|C_p^q| \leq \sum_{i=0}^{k-1} |C_i|$, implying

$$\delta(p, q) = \frac{|C_p^q|}{|pq|} \leq \frac{\sum_{i=0}^{k-1} |C_i|}{\sum_{i=0}^{k-1} |p_i p_{i+1}|}$$

   - Due to the fact (if $a_i/b_i \leq q$ for all i, $\sum_i a_i / \sum_i b_i \leq q$),

$$\delta(p, q) \leq \max_{0 \leq i \leq k-1} \frac{|C_i|}{|p_i p_{i+1}|} = \max_{0 \leq i \leq k-1} \delta(p_i, p_{i+1}).$$

2. p or q is a vertex
   - If p or q is a vertex, we are done.
   - Otherwise, we can move p and q simultaneously until the new segment $p'q'$ hit a vertex r.
     - If $r = p'$ or $r = q'$, we are done.
     - otherwise, either $\delta_C = \delta(r, p')$ or $\delta_C = \delta(r, q')$ such that we can argue as above.

All the co-visible vertex-edge pairs of a chain can be computed in time linear to their number, while their number is still quadratic.

Lemma 4
Let p, r, q, s be points on C that appear in that order, and assume pq and rs are two segment crossing each other. Then

$$\min(\delta(p,q),\delta(r,s)) < \max(\delta(r,q),\delta(,p,s)).$$

It is the same if the points appear in order p, r, s, q on C.

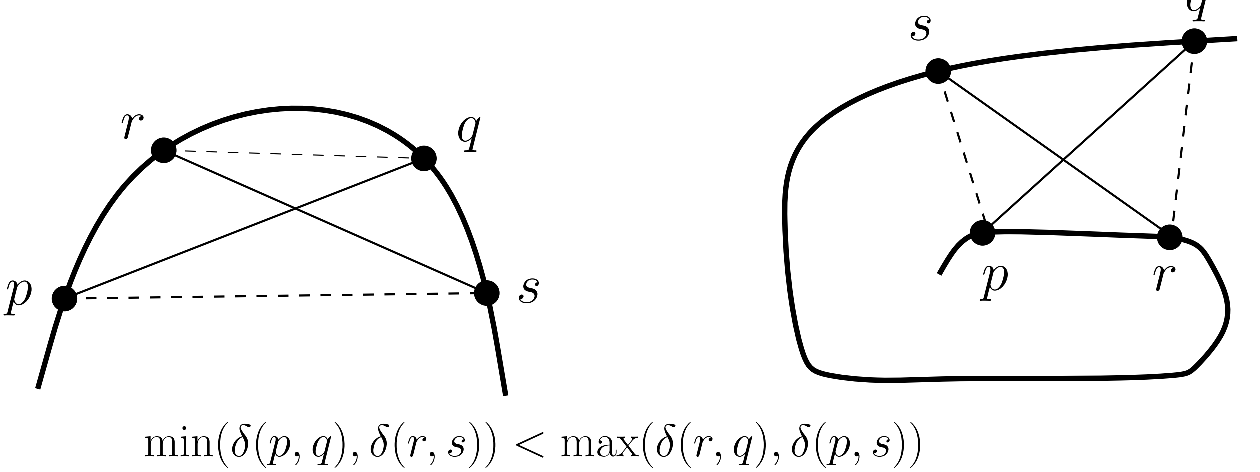
Proof

- W.lo.g., assume that $\delta(p,q) \le \delta(r,s)$ and $\delta(p,q) \ge \delta(r,q)$.
- By definition, $|C_p^q||rs| \le |C_r^s||pq|$ and $|C_p^q||rq| \ge |C_r^q||pq|$.
- We have to show $\delta(p,q) < \delta(p,s)$.
- By the triangle inequality,

  $$|ps| + |rq| < |pq| + |rs|.(pq \text{ and } rs \text{ cross each other.})$$

- $$|C_p^q|(|ps| + |rq|) < |C_p^q|(|pq| + |rs|) \le |C_p^q||pq| + |C_r^s||pq|$$
  $$= (C_p^q + C_r^s)|pq| = (C_p^s + C_r^q)|pq| \le |C_s^p||pq| + |C_p^q||pq|$$
- $|C_p^q||ps| < |C_p^s||pq| \to \delta(p,q) < \delta(p,s)$

Lemma 5
Let (p,q) and (r,s) be two vertex-edge cuts that attain the maximum detour δ_C. Then the segments pq and rs do not cross. Consequently there are only $O(n)$ such cuts altogether.



$\min(\delta(p,q),\delta(r,s)) < \max(\delta(r,q),\delta(p,s))$

Proof

- (p, q) and (r, s) are co-visible.
- If (p, q) and (r, s) are crossing, C will visit p, q, r, s in one of the two ways depicted in the figure.
- By Lemma 4, we would obtain a contradiction to the maximality of the detours $\delta(p, q)$ and $\delta(r, s)$.
- Finally, By Euler's formula, ther can be only $O(n)$ non-crossing segments stemming from vertex-edge cuts.

Summary

1. Let V be the set of vertices in the polygonal C, and let $\kappa \geq 1$. There is a pair $(p, q) \in C \times C$ so that $\delta(p, q) > \kappa$ if and only if there is pair $(p', q') \in C \times V$ so that $\delta(p', q') > \kappa$ and p' is visible from q'
2. Assume that the detour contains a local maximum at two points, q, q', that are interior poins of edges e, e' of C, correspondingly. Then the line segment qq' forms the same angle with e and e', and the detour of q, q' does not change as both point move, at the same speed, along their corresponding edges.
3. Let q, q' be two points on C, and assume that the line segment connecting them contains a third point, r, of C. Then $\max\{\delta(q, r), \delta(r, q')\} \geq \delta(q, q')$. Moreover, if the equality holds, then $\delta(q, r) = \delta(r, q') = \delta(q, q')$.

Reference:
A Ebbers-Baumann, R. Klein, E. Langetepe, and A. Lingas. A fast algorithm for approximating the detour of a polygonal chain. Computational Geometry: Theory and Applications, vol 27, pp. 123–134.