Voronoi Diagram and Delaunay Triangulation

Chih-Hung Liu

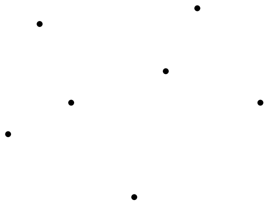
October 21

Outline

1. Voronoi Diagrams and Delaunay Triangulations
   - Properties and Duality
2. 3D geometric transformation

Voronoi Diagram

- Given a set S of n point sites, Voronoi Diagram $V(S)$ is a planar subdivision



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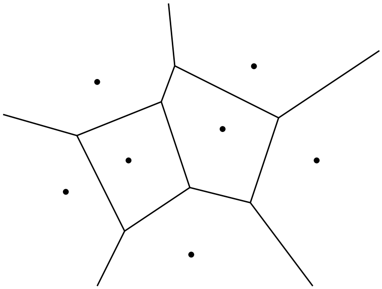
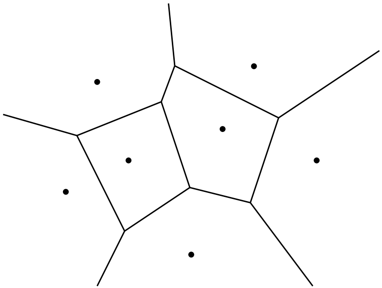


Voronoi Diagram

- Given a set S of n point sites, Voronoi Diagram $V(S)$ is a planar subdivision
  1. Each region contains exactly one site $p \in S$ and is denoted by $\mathrm{VR}(p, S)$.



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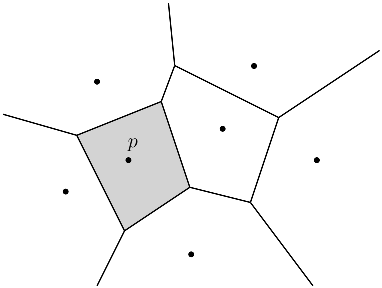

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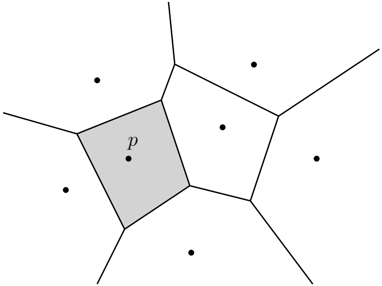

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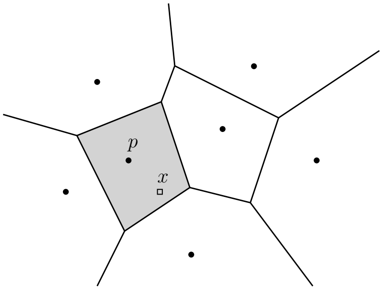

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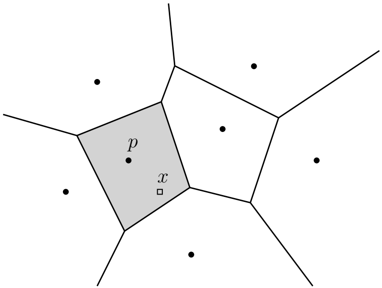
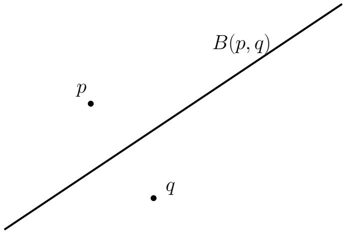
  2. For each point $x \in \mathrm{VR}(p, S)$, p is its closest site in S.
- $\mathrm{VR}(p, S)$ is the locus of points closer to p than any other site.



Voronoi Region

- Bisector **$B(p,q)$** $= \{x \in R^2 \mid d(x,p) = d(x,q)\}$.



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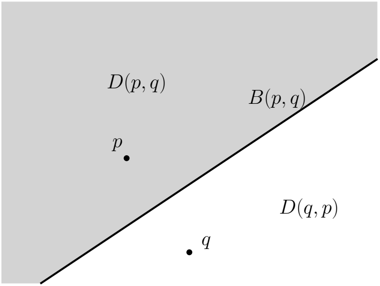
  - Two half-planes $D(p,q)$ and $D(q,p)$ separated by $B(p,q)$.



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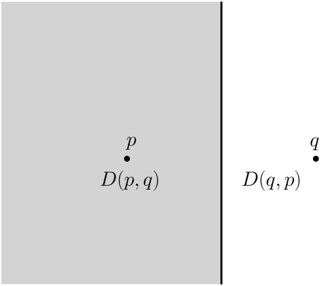

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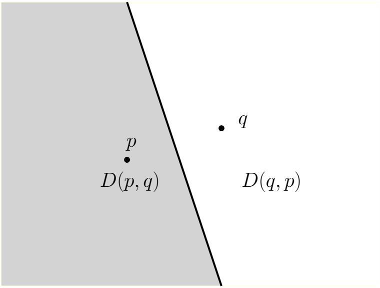

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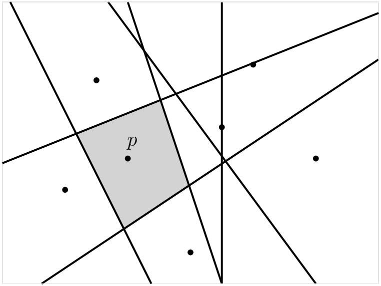
  - Two half-planes $D(p,q)$ and $D(q,p)$ separated by $B(p,q)$.
- $$\mathrm{VR}(p,S) = \bigcap_{q \in S, q \neq p} D(p,q).$$



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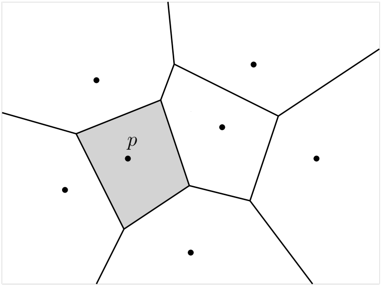
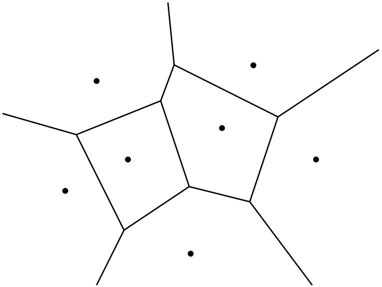


Voronoi Edge and Vertex

- Voronoi Edge
  - Common intersection between two adjacent Voronoi regions $VR(p, S)$ and $VR(q, S)$



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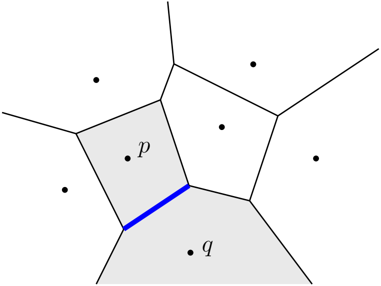

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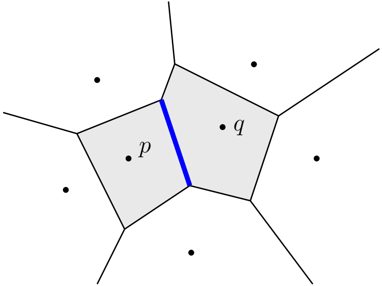

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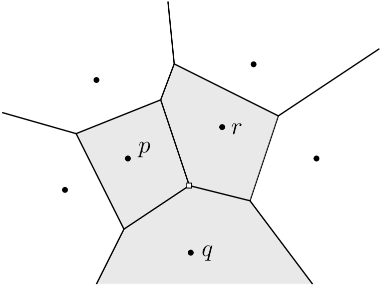
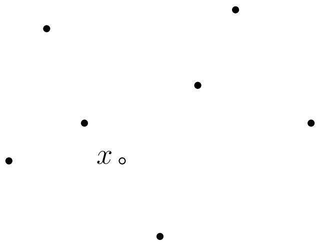
  - Common intersection among more than two Voronoi regions $\text{VR}(p, S)$, $\text{VR}(q, S)$, $\text{VR}(r, S)$, and so on.



Growing Circle

- Grow a circle from a point x on the plane



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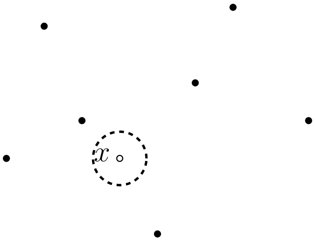

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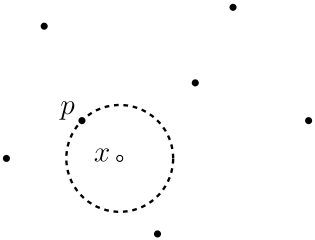

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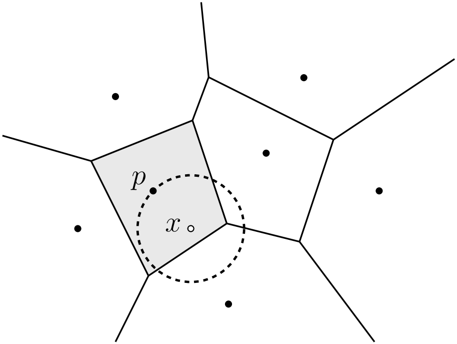

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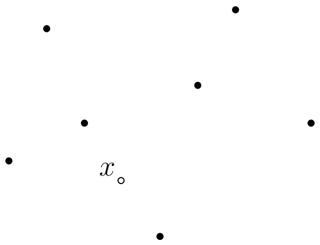

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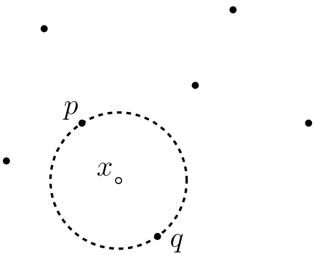

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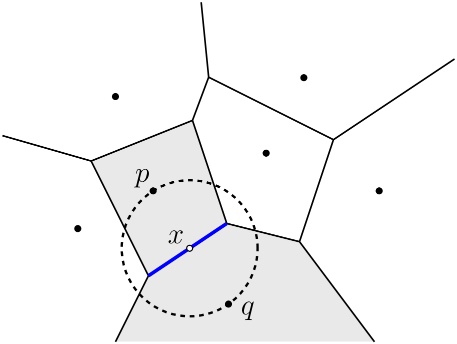

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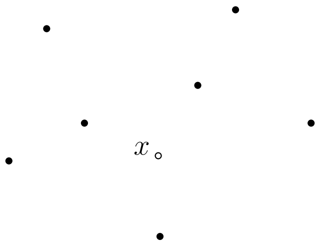

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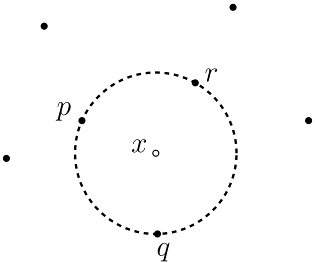

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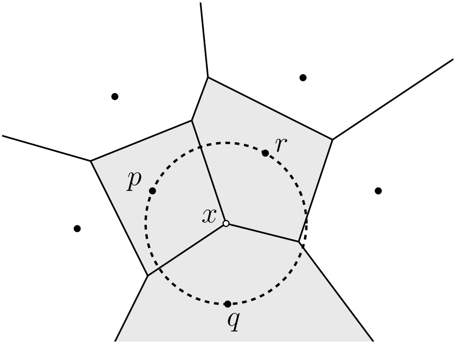
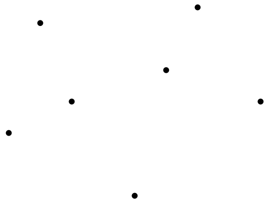


Wavefront Model (Growth Model)

- Grow circles from $\forall p \in S$ at unit speed



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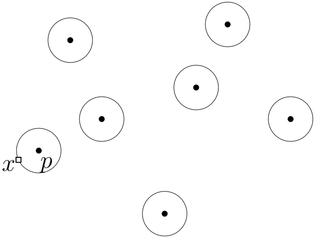

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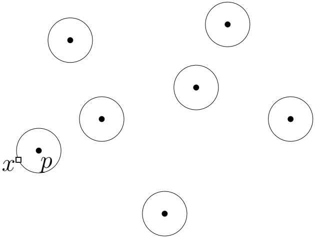

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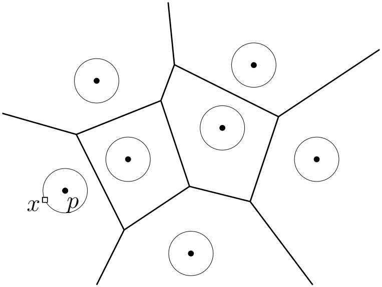

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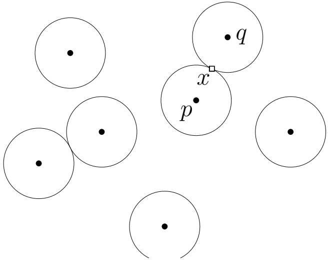

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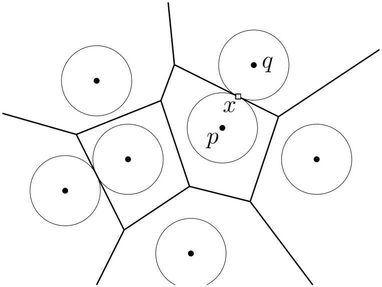

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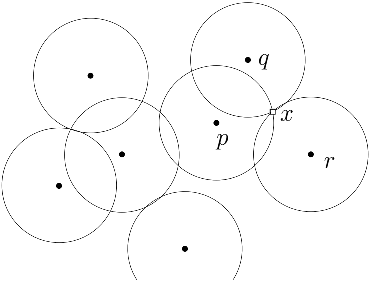

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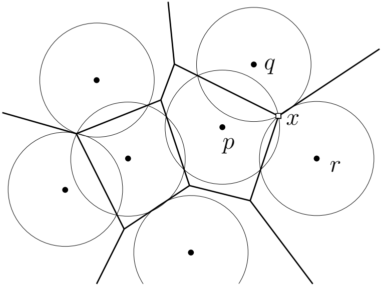
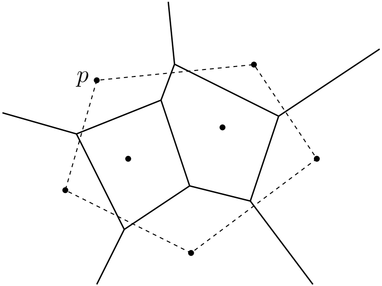


Unbounded Region

- VR(p, S) is unbounded if and only if p is a vertex of the convex hull of S.



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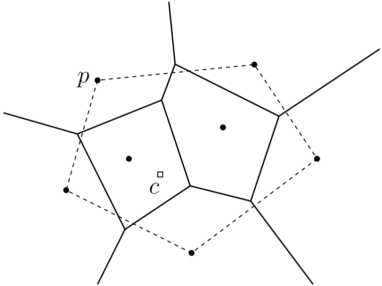

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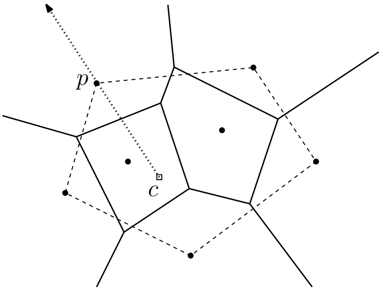

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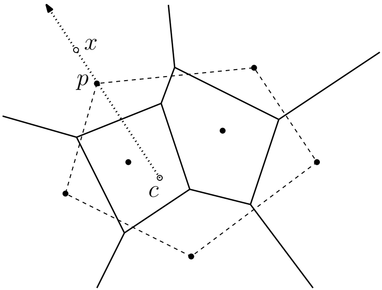
  - Shoot a ray $\overrightarrow{cp}$ from c to p
  - For any point $x \in \overrightarrow{cp} \setminus \overline{cp}$, x belongs to VR(p, S)



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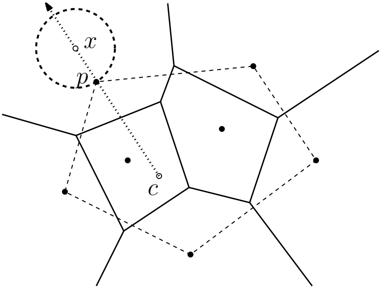

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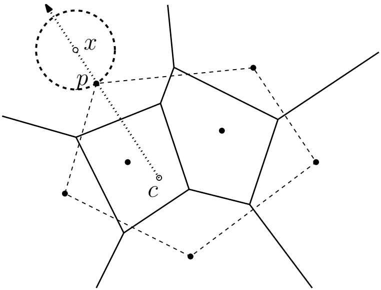
  - Shoot a ray $\overrightarrow{cp}$ from c to p
  - For any point $x \in \overrightarrow{cp} \setminus \overline{cp}$, x belongs to VR(p, S)
  - $\overrightarrow{cp}$ extends to the infinity.



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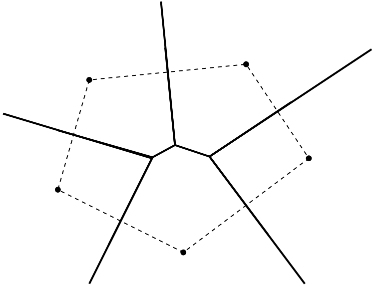
  - For any point $x \in \overrightarrow{cp} \setminus \overline{cp}$, x belongs to VR(p, S)
  - $\overrightarrow{cp}$ extends to the infinity.
- If S is in convex position, $V(S)$ is a tree.



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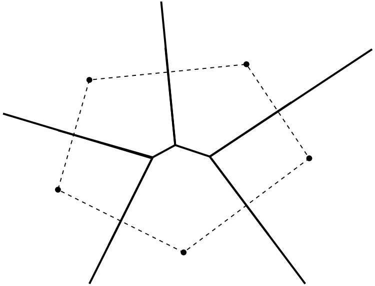
- If S is in convex position, $V(S)$ is a tree.
- An unbounded Voronoi edge corresponds to a hull edge.



Voronoi Diagram (Mathematic Definition)

- Voronoi Diagram $V(S)$

$$V(S) = R^2 \setminus (\bigcup_{p \in S} \text{VR}(p, S)) = \bigcup_{p \in S} \partial\text{VR}(p, S)$$

  - $\partial\text{VR}(p, S)$ is the boundary of $\text{VR}(p, S)$
    - $\partial\text{VR}(p, S) \not\subset \text{VR}(p, S)$
  - $V(S)$ is the union of all the Voronoi edges
- Voronoi Edge e between $\text{VR}(p, S)$ and $\text{VR}(q, S)$

$$e = \partial\text{VR}(p, S) \cap \partial\text{VR}(q, S)$$

- Voronoi Vertex v among $\text{VR}(p, S)$, $\text{VR}(q, S)$, and $\text{VR}(r, S)$

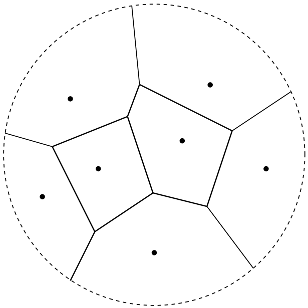
$$v = \partial\text{VR}(p, S) \cap \partial\text{VR}(q, S) \cap \partial\text{VR}(r, S)$$

Complexity of $V(S)$

Theorem

$V(S)$ has $O(n)$ edges and vertices. The average number of edges of a Voronoi region is less than 6.

- Add a large curve Γ
  - Γ only passes through unbounded edges of $V(S)$
  - Cut unbounded pieces outside Γ
  - One additional face and several edges and vertices.



Complexity of $V(S)$

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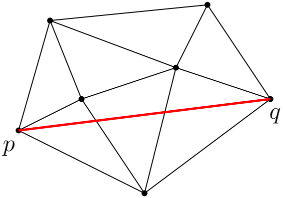
- Euler's Polyhedron Formula: $v - e + f = 1 + c$
  - v: # of vertices, e: # of edges, f: # of faces, and c: # number of connected components.
- An edge has two endpoints, and a vertex is incident to at least three edges.
  - $3v \leq 2e \rightarrow v \leq 2e/3$
- $f = n + 1$ and $c = 1$
  - $v = 1 + c + e - f = e + 1 - n \leq 2e/3 \rightarrow e \leq 3n - 3$
  - $e = v + f - 1 - c = v + n - 1 \geq 3v/2 \rightarrow v \leq 2n - 2$
- Average number of edges of a region $\leq (6n - 6)/n < 6$

Triangulation

Definition

Given a set *S* of points on the plane, a triangulation is maximal collection of non-crossing line segments among *S*.



Crossing ($\overline{pq}$)

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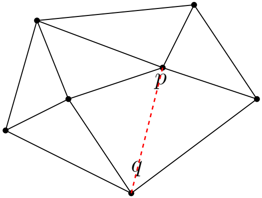

Not Maximal ($\overline{pq}$ is allowable)

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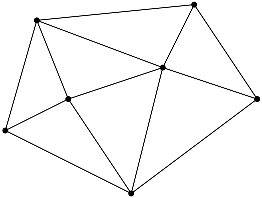
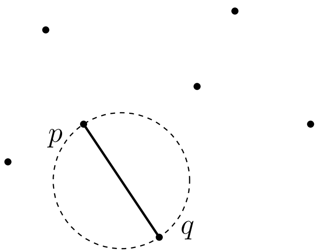


Triangulation

Delaunay Edge

Definition

An edge $\overline{pq}$ is called **Delaunay** if there exists a circle passing through p and q and containing no other point in its interior.



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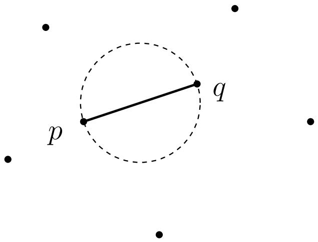

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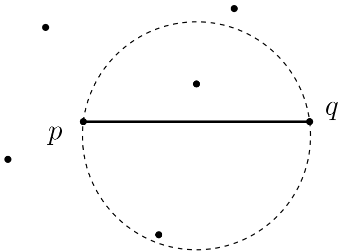
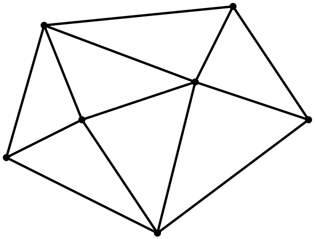


$\overline{pq}$ is **NOT** Delaunay

Delaunay Triangulation

Definition

A **Delaunay Triangulation** is a triangulation whose edges are all Delaunay.



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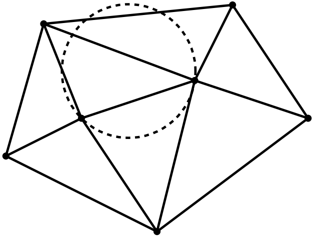

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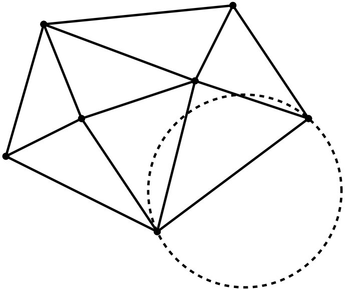

Delaunay Triangulation

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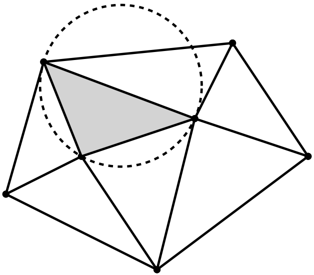
- For each face, there exists a circle passing all its vertices and containing no other point.



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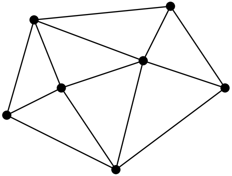
- There is a unique Delaunay triangulation.

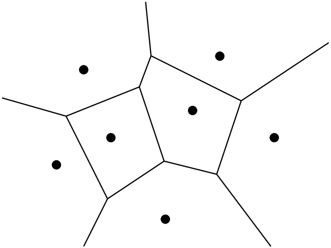
Duality

Theorem

Under the general position assumption, the Delaunay triangulation is a dual graph of the Voronoi diagram.

- A site $p \leftrightarrow$ a Voronoi region $VR(p, S)$





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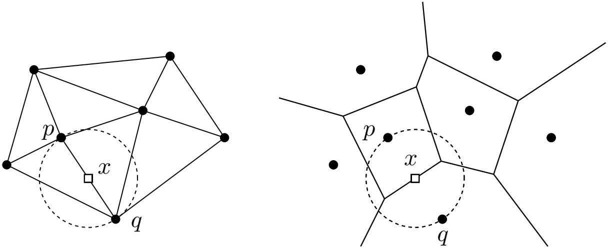

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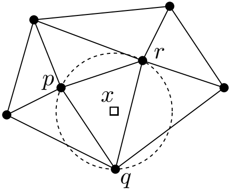
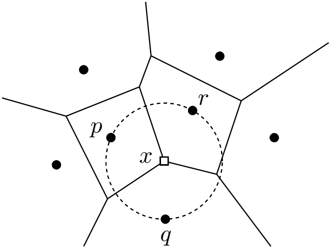
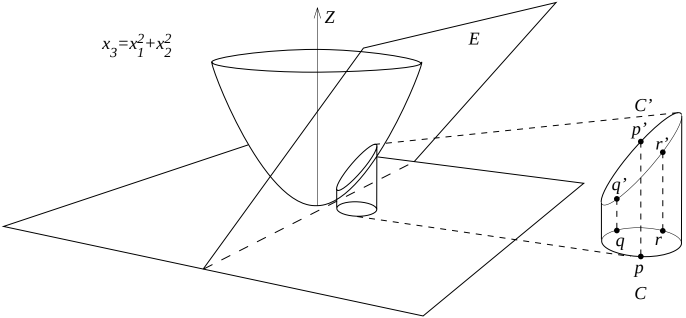




Geometric Transformation from 2D to 3D

- A paraboloid $P = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = x_3\}$ in 3D
- For a point $x = (x_1, x_2)$ in 2D, $x' = (x_1, x_2, x_1^2 + x_2^2)$ is its lifted image in 3D
  - $x' \leftarrow$ vertical projection from x to P
- For a set A of points in 2D, its lifted image $A' = \{x' = (x_1, x_2, x_1^2 + x_2^2) \mid x = (x_1, x_2) \in A\}$



Circle in 2D $\leftrightarrow$ Planar Curve in P

Lemma

Let C be a circle in the plane. Then C' is a planar curve on the paraboloid P

- C is given by $r^2 = (x_1 - c_1)^2 + (x_2 - c_2)^2$
  - $r^2 = x_1^2 + x_2^2 - 2x_1c_1 - 2x_2c_2 + c_1^2 + c_2^2$
- C' satisfies $x_1^2 + x_2^2 = x_3$
- Substituting $x_1^2 + x_2^2$ by x_3, we obtain a plane E

$$x_3 - 2x_1c_1 - 2x_2c_2 + c_1^2 + c_2^2 - r^2 = 0$$

- $C' = P \cap E$
- Intersection between E and P is a planar curve

Lower Convex Hull

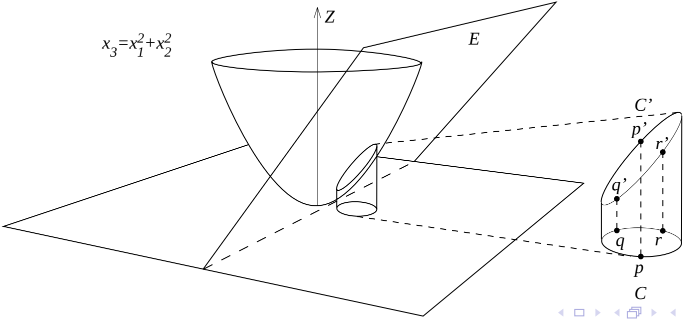
- S' on $P \rightarrow S'$ in convex position
- Each point of S' is a vertex of $conv(S')$
- Lower convex hull $lconv(S')$ of S' is the part of $conv(S')$ visible from $x_3 = -\infty$

Duality between $\mathrm{DT}(S)$ and $lconv(S')$ (1)

Theorem

The Delaunay triangulation $\mathrm{DT}(S)$ equals to the vertical projection onto the $x_1 x_2$-plane of the lower convex hull of S'

- $p, q, r \in S$. C: circumcircle of p, q, r
- C' lies on a plane E defined by p', q', r'
- a point x inside $C \leftrightarrow$ lifted image x' below E



Duality between $\mathrm{DT}(S)$ and $lconv(S')$ (2)

Theorem

The Delaunay triangulation $\mathrm{DT}(S)$ equals to the vertical projection onto the $x_1 x_2$-plane of the lower convex hull of S'

- p, q, r defines a triangle of $\mathrm{DT}(S)$
  $\leftrightarrow$ no point of S in C defined by p, q, r
  $\leftrightarrow$ no point of S' below E defined by p', q', r'
  $\leftrightarrow$ p', q', r' defines a facet of $lconv(S')$
- Computing a convex hull in 3D takes $O(n \log n)$ time
  - $V(S)$ in $O(n \log n)$ time

Another Viewpoint of paraboloid

- For each $s = (s_1, s_2) \in S$, a paraboloid

$$P_s = \{(x_1, x_2, x_3) \mid x_3 = (x_1 - s_1)^2 + (x_2 - s_2)^2\}$$

  - For each $x = (\sigma_1, \sigma_2)$ in $x_1 x_2$ plane, vertical distance from x to P_s is $d(x, s)^2$
- Opaque and of pairwise different colors
- Looking from $x_3 = -\infty$ upward $\rightarrow V(S)$
- Vertical from x upward first hits $P_s \rightarrow x \in \mathrm{VR}(p, S)$
- $P_s \cap P_t \rightarrow B(s, t)$
- Lower envelope of $\bigcup_{s \in S} P_s \rightarrow V(S)$

Wavefront model revisited

- $P_s = \{(x_1, x_2, x_3) | x_3 = f((x_1 - s_1)^2 + (x_2 - s_2)^2)\}$
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- Expanding circles C_s from sites $s \in S$ at equal unit speed
  - time t = radius r
  - $r^2 = (x_1 - s_1)^2 + (x_2 - s_2)^2$
  - $x_3 = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2}$ = radius = time

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- x first hit by $C_s \leftrightarrow$ upward vertical projection from x first hit P_s

Thank You!!