Recall WSPD

* Build recursively split tree $T(s)$ in $O(n)$ time.

* For each $u$ in $T(s)$, call FindPairs $(v, w)$.

  - if $v, w$ w.s. $s$, return.
  - else if $v$ has longer bounding box than $w$, then:
    - FindPairs $(v, w)$, FindPairs $(w, v)$.

$\Rightarrow$ WSPD consisting of w.s. pairs $A_i, B_i$ s.t.

$$S^2 \subseteq \bigcup_{i=1}^{n} A_i \times B_i \cup \bigcup_{i=1}^{n} B_i \times A_i$$

Running time:

* # Calls to FindPairs $\in O(n)$.

Proof: for each node $u$ in $T(s)$:

- Recursion tree $(v, w)$.

- All leaves of these trees $\Rightarrow$ well-separated pairs found (each only once).

* $m \in \Theta(n)$ if dimension $d$ and separation $s$ are fixed.

Main proof idea: show there are only $O(1)$ sets $B_i$ s.t. pair $(A_i, B_i)$ is reported, for each $A_i$ (only $O(n)$ many).

Class: $(A_i, B_i)$ reported $\Rightarrow B_i, B_j$ separated by hyperplane, bounding boxes disjoint.

$(\forall$ nodes $v, w$ in $T(s): s_v \neq s_w$ or $s_w \leq s_v$ or sep.)
Moreover: pair \((A, B_i)\) separate, pair \((\pi(A), B_i)\) not well separated (or vice versa)

\[ \Downarrow \text{father} \]

\[ A_i \xrightarrow{\pi(A)} B_i \]

\[ \Downarrow \]

\[ \Rightarrow \text{the boxes of all } B_i \text{ are close to each other} \]

\[ \Rightarrow \text{now use packing argument} \]

**Theorem** WSPD of \(n\) points in \(\mathbb{R}^d\): \(O(n \log n)\) time \(O(n)\) space

**Applications**

- Closest pair in time \(O(m) \subset O(n)\)

for each \(p \in S\), all \(k\) nearest neighbors: \(O(n \log n + nk)\)

\((V(s)\) can do this only for \(k = 1\))

- Post office: give arbitrary \(q \in \mathbb{R}^d\), report nearest \(p \in S\)

  - Case of dimensionality
  - Only approximate solutions known
  - Report \(p'\) s.t. \(|q - p'| \leq (1 + \epsilon)|q - p|\)
  - Using dynamic structures
main WSPD application: construction of good geometric networks connecting points in S

good: low dilation, \( \Delta(N) = \min_{\text{edges}} \frac{1}{\lambda(N)} \)

few edges

complete graph: dilation 1, optimal

(spanning tree: can be \( \Omega(n) \))

Small miracle: Construct WSPD for \( S \) wrt \( S \times Y : O(|\log n|) \) for each w.s. pair \( A_i, B_i : \)

- Pick \( p_i \in A_i \), \( q_i \in B_i \)
- Add edge \( p_i q_i \)

\( \Rightarrow \quad O(m) \leq O(n) \) edges (by previous Theorem)

dilation \( \leq \frac{n+4}{n-4} \) as \( n \to \infty \) (by induction on rank \( \lambda(1+q) \))

induct. basis

\( p q \) closest pair \( \Rightarrow \quad S_i : A_i = \{ p \} \quad B_i = \{ q \} \)

induct. step

\( \frac{|p|}{|q'|} < \frac{2}{3} \) \( |q'| \)

\( |p| < \left( 1 + \frac{4}{3} \right) |q'| \)