Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Cont. Geometric Firefighting – Lower Bound and FF Curve

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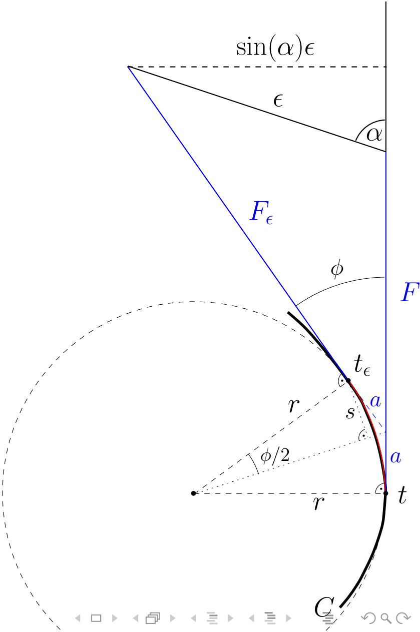
Helping Lemmata

Lemma 61: $\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$

Lemma 62: String of length F is tangent to point t on smooth curve C. End of string moves distance ϵ in direction α. For the curve length $C_t^{t_\epsilon}$ between t and the new tangent point, t_ϵ, we have

$$\lim_{\epsilon \to 0} \frac{C_t^{t_\epsilon}}{\epsilon} = \frac{r \sin \alpha}{F}$$

where r denotes radius of osculating circle at t.



Helping Lemmata

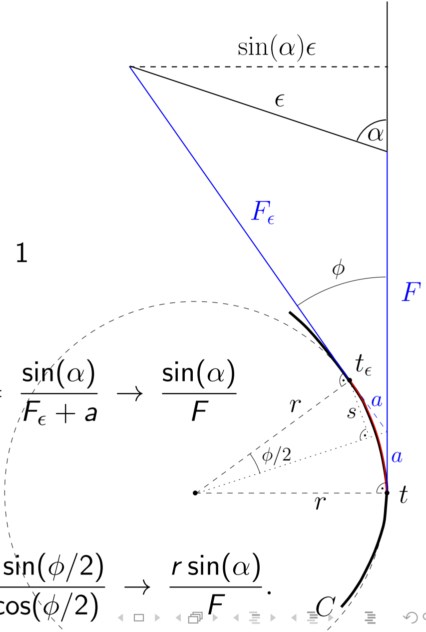
$r\ \sin(\phi/2) = s = a\ \cos(\phi/2)$
gives $a = r\ \tan(\phi/2)$

$2a$ approximates $c := C_t^{t_\epsilon}$:

$$\frac{c}{2a} = \frac{r\,\phi}{2r\ \tan(\phi/2)} \approx \cos^2(\phi/2) \to 1$$

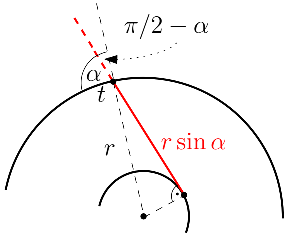
$$\frac{\epsilon\ \sin(\alpha)}{\sin(\phi)} = \frac{F_\epsilon + a}{\sin(\pi/2)} \text{ gives } \frac{\sin(\phi)}{\epsilon} = \frac{\sin(\alpha)}{F_\epsilon + a} \to \frac{\sin(\alpha)}{F}$$

$\sin(\phi/2)/\epsilon \ \to\ \sin(\alpha)/(2F)$



$$\frac{C_t^{t_\epsilon}}{\epsilon} = \frac{c}{2a}\,\frac{2a}{\epsilon} \approx \frac{2r\ \tan(\phi/2)}{\epsilon} = \frac{2r\ \sin(\phi/2)}{\epsilon\ \cos(\phi/2)} \to \frac{r\sin(\alpha)}{F}.$$

Helping Lemmata



Lemma 63: Let t be point on smooth curve C, osculating circle at t has radius r. Lines L_s resulting from turning the normal at points s by angle of $\pi/2 - \alpha$. Limit intersection point of normals with L_t has distance $\sin\alpha\, r$ to t.

Helping Lemmata

Lemma 61: $\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$

- Curve L_{j-1}, F_j, F_{j-1} depending on L_j, which depends on l
- Lemma 62: $\frac{L'_{j-1}(L_j)}{L'_j(L_j)} = L'_{j-1}(L_j) = \frac{r \sin \alpha}{F_j(L_j)}$
- FF Curve: Normal turned by $\pi/2 - \alpha$, tangents to previous coil
- Lemma 63: $F_{j-1}(L_j) = r \sin \alpha$
- Substitute L_j with $L_j(l)$ (derivatives cancel out!):

$$\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$$

Build and solve differential equations

1. $\frac{F_j(l)}{F_0(l)} = \frac{L_j'(l)}{l'} = L_j'(l)$ (Lemma 61/Multiplication)

2. $F_j'(l) + L_{j-1}'(l) = \cos\alpha \, L_j'(l)$ (Lemma 60/Derivatives)

1+2 = Lin. Diff. Eq. $\quad F_j'(l) - \frac{\cos(\alpha)}{F_0(l)} F_j(l) = -\frac{F_{j-1}(l)}{F_0(l)}$.

Textbook solution of $y'(x) + f(x)y(x) = g(x)$

$$y(x) = \exp(-a(x)) \left(\int g(t) \exp(a(t)) \, \mathrm{d}t + \kappa \right)$$

With $a = \int f$ and constant κ

Build and solve differential equations

$$F_j'(l) \;-\; \frac{\cos(\alpha)}{F_0(l)}\,F_j(l) \;=\; -\frac{F_{j-1}(l)}{F_0(l)}\,.$$

Textbook solution of $y'(x) + f(x)y(x) = g(x)$

$$y(x) = \exp(-a(x))\left(\int g(t)\exp(a(t))\,\mathrm{d}t + \kappa\right)$$

With $a = \int f$ and constant κ

$$a(l) \;=\; \int -\frac{\cos(\alpha)}{A + \cos(\alpha)\,l} \;=\; -\ln(F_0(l))$$

because of $F_0(l) = A + \cos(\alpha)\,l$, and we obtain

$$F_j(l) \;=\; F_0(l)\Big(\kappa_j \;-\; \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big).$$

Build and solve differential equations

For $l \in [0, l_1]$, F-linkages:

$$F_j(l) \;=\; F_0(l)\Big(\kappa_j \;-\; \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big). \tag{1}$$

Same arguments and $l \in [l_1, l_2]$, ϕ-linkages:

$$\phi_j(l) \;=\; \phi_0(l)\Big(\lambda_j \;-\; \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)}\,\mathrm{d}t\Big). \tag{2}$$

Successively resolve the constants κ_j, λ_j by:

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \end{aligned}$$

Build and solve differential equations

$$F_j(l) \;=\; F_0(l)\Big(\kappa_j \;-\; \int \frac{F_{j-1}(t)}{F_0^2(t)}\,\mathrm{d}t\Big). \tag{3}$$

$$\phi_j(l) \;=\; \phi_0(l)\Big(\lambda_j \;-\; \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)}\,\mathrm{d}t\Big). \tag{4}$$

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \end{aligned}$$

$F_{-1} = \phi_{-1} = 0,\; F_0(l) = A + l\cos\alpha,\; \kappa_0 = 1,\; \lambda_0 = 1$

Example: κ_1 with $\phi_0(l_2) = F_1(0)$ gives

$$\kappa_1 := \frac{\phi_0(l_2)}{F_0(0)} \;+\; \int \frac{F_0(t)}{F_0^2(t)}\mathrm{d}t|_{l=0}$$

Build and solve differential equations

In general:

$$\kappa_{j+1} := \frac{\phi_j(l_2)}{F_0(0)} + \int \frac{F_j(t)}{F_0^2(t)} \mathrm{d}t|_{l=0}$$

$$\lambda_{j+1} := \frac{F_{j+1}(l_1)}{\phi_0(l_1)} + \int \frac{\phi_j(t)}{\phi_0^2(t)} \mathrm{d}t|_{l=l_1}$$

so that

$$F_{j+1}(l) = F_0(l) \left(\frac{\phi_j(l_2)}{F_0(0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)} \, \mathrm{d}t \right),$$

$$\phi_{j+1}(l) = \phi_0(l) \left(\frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)} \, \mathrm{d}t \right),$$

$$\begin{aligned} F_{j+1}(l_1) &= \phi_{j+1}(l_1) \\ F_{j+1}(0) &= \phi_j(l_2) \end{aligned}$$

Final formulas:

$$F_{j+1}(l) \;=\; F_0(l)\left(\frac{\phi_j(l_2)}{F_0(0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)}\,\mathrm{d}t\right),$$

$$\phi_{j+1}(l) \;=\; \phi_0(l)\left(\frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)}\,\mathrm{d}t\right),$$

For simplicity, let us write

$$G_j(l) := \frac{F_j(l)}{F_0(l)} \;\text{ and }\; \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)},$$

which leads to

$$\begin{aligned} G_{j+1}(l) &= \frac{\phi_0(l_2)}{F_0(0)}\,\chi_j(l_2) - \int_0^l \frac{G_j(t)}{F_0(t)}\,\mathrm{d}t \\ \chi_{j+1}(l) &= \frac{F_0(l_1)}{\phi_0(l_1)}\,G_{j+1}(l_1) - \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)}\,\mathrm{d}t. \end{aligned}$$

Final formulas:

$$G_j(l) := \frac{F_j(l)}{F_0(l)} \text{ and } \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)},$$

which leads to

$$\begin{aligned} G_{j+1}(l) &= \frac{\phi_0(l_2)}{F_0(0)}\,\chi_j(l_2) - \int_0^l \frac{G_j(t)}{F_0(t)}\,\mathrm{d}t \\ \chi_{j+1}(l) &= \frac{F_0(l_1)}{\phi_0(l_1)}\,G_{j+1}(l_1) - \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)}\,\mathrm{d}t. \end{aligned}$$

Lemma 64: The curve encloses the fire if and only if there exists an index j such that $F_j(l_1) \leq 0$ holds.

Let integrals disappear:

$$
\begin{aligned}
G_{j+1}(l) &= \frac{\phi_0(l_2)}{F_0(0)}\,\chi_j(l_2) \;-\; \int_0^l \frac{G_j(t)}{F_0(t)}\,\mathrm{d}t \\
\chi_{j+1}(l) &= \frac{F_0(l_1)}{\phi_0(l_1)}\,G_{j+1}(l_1) \;-\; \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)}\,\mathrm{d}t.
\end{aligned}
$$

Iterated insertion: $G_0(l) = 1$, $G_1(l) = \frac{\phi_0(l_2)}{F_0(0)}\,\chi_0(l_2) \;-\; \int_0^l \frac{1}{F_0(t)}\,\mathrm{d}t$

$$
G_2(l) = \frac{\phi_0(l_2)}{F_0(0)}\,\chi_1(l_2) \;-\; \int_0^l \frac{\frac{\phi_0(l_2)}{F_0(0)}\,\chi_0(l_2) \;-\; \int_0^t \frac{1}{F_0(t_1)}\,\mathrm{d}t_1}{F_0(t)}\,\mathrm{d}t =
$$

$$
\frac{\phi_0(l_2)}{F_0(0)}\left(\chi_1(l_2) \;-\chi_0(l_2)\int_0^l \frac{1}{F_0(t)}\,\mathrm{d}t\right)+\int_0^l \frac{1}{F_0(t)}\int_0^t \frac{1}{F_0(t_1)}\,\mathrm{d}t_1\mathrm{d}t
$$

Iterated insertion!

$$G_{j+1}(l) \;=\; \tfrac{\phi_0(l_2)}{F_0(0)}\,\chi_j(l_2) \;-\; \int_0^l \tfrac{G_j(t)}{F_0(t)}\,\mathrm{d}t,\; G_2(l) =$$

$$\frac{\phi_0(l_2)}{F_0(0)}\,\left(\chi_1(l_2)\;-\chi_0(l_2)\int_0^l \frac{1}{F_0(t)}\,\mathrm{d}t\right)+\int_0^l \frac{1}{F_0(t)}\,\int_0^t \frac{1}{F_0(t_1)}\,\mathrm{d}t_1\mathrm{d}t$$

$G_3(l) =$?? Blackboard!

$$I_n(x_n) \;:=\; \int_0^{x_n} \frac{1}{F_0(x_{n-1})}\int_0^{x_{n-1}} \frac{1}{F_0(x_{n-2})}\cdots\int_0^{x_1}\frac{1}{F_0(x_0)}\,\mathrm{d}x_0\ldots\,\mathrm{d}x_{n-1}.$$

By induction on n (Exercise!)

$$I_n(x_n) \;=\; \frac{1}{n!}\,\frac{1}{\cos^n\alpha}\,\left(\ln\left(\frac{A+(\cos\alpha)x_n}{A}\right)\right)^n$$

For $F_0(x) = A + \cos\alpha\, x$

Iterated insertion!

$$I_n(x_n) \;:=\; \int_0^{x_n} \frac{1}{F_0(x_{n-1})} \int_0^{x_{n-1}} \frac{1}{F_0(x_{n-2})} \cdots \int_0^{x_1} \frac{1}{F_0(x_0)} \,\mathrm{d}x_0 \ldots \,\mathrm{d}x_{n-1}.$$

$$I_n(x_n) \;=\; \frac{1}{n!}\,\frac{1}{\cos^n \alpha}\,\left(\ln\left(\frac{A + (\cos\alpha)x_n}{A}\right)\right)^n$$

$$G_{j+1}(l) \;=\; \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^{j} (-1)^\nu \, I_\nu(l)\, \chi_{j-\nu}(l_2) \;+\; (-1)^{j+1} I_{j+1}(l).$$

$G_{j+1}(l_1)$ for $l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$: $\ln\left(\frac{A+(\cos\alpha)l_1}{A}\right) = 2\pi \cot\alpha$

Final Formula!

$$G_{j+1}(l) \;=\; \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^{j} (-1)^{\nu}\, I_{\nu}(l)\, \chi_{j-\nu}(l_2) \;+\; (-1)^{j+1} I_{j+1}(l).$$

$$I_n(l_1) \;=\; \frac{1}{n!} \left(\frac{2\pi}{\sin\alpha} \right)^{n}$$

$$G_j(l_1) \;=\; \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^{\nu}}{\nu!} \left(\frac{2\pi}{\sin\alpha} \right)^{\nu} \chi_{j-1-\nu}(l_2)$$

whith $\chi_{-1}(l_2) := \frac{F_0(0)}{\phi_0(l_2)}$.

Final formulas, substitute again:

$$G_j(l) := \frac{F_j(l)}{F_0(l)} \text{ and } \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)},$$

$$G_j(l_1) = \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{2\pi}{\sin\alpha}\right)^\nu \chi_{j-1-\nu}(l_2)$$

$$F_j(l_1) = \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{2\pi}{\sin\alpha}\right)^\nu \phi_{j-1-\nu}(l_2)$$

where $\phi_{-1}(l_2) := F_0(0)$

The same reasoning for ϕ:

$$\phi_j(l_2) \;=\; \frac{\phi_0(l_2)}{\phi_0(l_1)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{\alpha}{\sin\alpha}\right)^\nu \hat{F}_{j-\nu}(l_1)$$

where $\hat{F}_0(l_1) := \phi_0(l_1)$ and $\hat{F}_{i+1}(l_1) := F_{i+1}(l_1)$.

$$F_j(l_1) \;=\; \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{2\pi}{\sin\alpha}\right)^\nu \phi_{j-1-\nu}(l_2)$$

Consider as coefficients of power series:

$$F(Z) := \sum_{j=0}^{\infty} F_j\, Z^j \quad \text{and} \quad \phi(Z) := \sum_{j=0}^{\infty} \phi_j\, Z^j$$

where $F_j := F_j(l_1)$ and $\phi_j := \phi_j(l_2)$

With $e^W = \sum_{j=0}^{\infty} \frac{W^j}{j!}$ and $W = -\frac{2\pi}{\sin\alpha}$

$$F_j(l_1) \;=\; \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{1}{\nu!} \left(\frac{-2\pi}{\sin\alpha}\right)^{\nu} \phi_{j-1-\nu}(l_2)$$

$$F(Z) \;=\; \frac{F_0}{F_0(0)}\, e^{-\frac{2\pi}{\sin\alpha} Z} \left(Z\,\phi(Z) + F_0(0)\right),$$

Consider as coefficients of power series:

$$F(Z) := \sum_{j=0}^{\infty} F_j\, Z^j \quad \text{and} \quad \phi(Z) := \sum_{j=0}^{\infty} \phi_j\, Z^j$$

where $F_j := F_j(l_1)$ and $\phi_j := \phi_j(l_2)$

$$F(Z) \;=\; \frac{F_0}{F_0(0)}\, e^{-\frac{2\pi}{\sin\alpha} Z} \left(Z\,\phi(Z) + F_0(0)\right)$$

Also

$$\phi(Z) \;=\; \frac{\phi_0}{\phi_0(l_1)}\, e^{-\frac{\alpha}{\sin\alpha} Z} \left(Z\,F(Z) - F_0 + \phi_0(l_1)\right)$$

Consider as coefficients of power series:

$$F(Z) \;=\; \frac{F_0}{F_0(0)}\, e^{-\frac{2\pi}{\sin\alpha} Z} \left(Z\,\phi(Z) + F_0(0)\right)$$

$$\phi(Z) \;=\; \frac{\phi_0}{\phi_0(l_1)}\, e^{-\frac{\alpha}{\sin\alpha} Z} \left(Z\,F(Z) - F_0 + \phi_0(l_1)\right)$$

Conclude:

$$\frac{F(Z)}{F_0} \;=\; \frac{e^{vZ} \;-\; r\,Z}{e^{wZ} \;-\; s\,Z},$$

where v, r, w, s are the following functions of α:

$$\begin{aligned} v \;&=\; \frac{\alpha}{\sin\alpha} \quad \text{and} \quad r \;=\; e^{\alpha\cot\alpha} \\ w \;&=\; \frac{2\pi+\alpha}{\sin\alpha} \quad \text{and} \quad s \;=\; e^{(2\pi+\alpha)\cot\alpha}. \end{aligned}$$

Analytic function in the complex plane:

$$\frac{F(Z)}{F_0} = \frac{e^{vZ} - rZ}{e^{wZ} - sZ},$$

$$\begin{aligned} v &= \frac{\alpha}{\sin\alpha} \quad \text{and} \quad r = e^{\alpha\cot\alpha} \\ w &= \frac{2\pi+\alpha}{\sin\alpha} \quad \text{and} \quad s = e^{(2\pi+\alpha)\cot\alpha}. \end{aligned}$$

Theorem (Pringsheim)

Let $H(Z) = \sum_{n=0}^{\infty} a_n Z^n$ be a power series with finite radius of convergence, R. If $H(Z)$ has only non-negative coefficients a_n, then point $Z = R$ is a singularity of $H(z)$.

Analytic function in the complex plane:

$$\frac{F(Z)}{F_0} = \frac{e^{vZ} - r\,Z}{e^{wZ} - s\,Z},$$

where v, r, w, s are the following functions of α:

$$\begin{aligned} v &= \frac{\alpha}{\sin\alpha} \quad \text{and} \quad r = e^{\alpha\cot\alpha} \\ w &= \frac{2\pi+\alpha}{\sin\alpha} \quad \text{and} \quad s = e^{(2\pi+\alpha)\cot\alpha}. \end{aligned}$$

Result from function theory!

Lemma

(Flajolet) For $s < ew$, equation

$$e^{wZ} - sZ.$$

has an infinite, discrete set of conjugate complex zeroes none of which are real.

Analytic function in the complex plane:

$$\frac{F(Z)}{F_0} = \frac{e^{vZ} - rZ}{e^{wZ} - sZ},$$

$$\begin{aligned} v &= \frac{\alpha}{\sin\alpha} \quad \text{and} \quad r = e^{\alpha\cot\alpha} \\ w &= \frac{2\pi+\alpha}{\sin\alpha} \quad \text{and} \quad s = e^{(2\pi+\alpha)\cot\alpha}. \end{aligned}$$

Lemma

For $s < ew$, function $F(Z)$ has an infinite, discrete set of complex poles none of which are real.

Exactly for $v_c = 2.6144\ldots$ does the equality $s/w = e$ hold. Hence, for $v > v_c$ we have $s < ew$

Upper bound by FollowFire

Theorem 59: FollowFire strategy is successful if $v > v_c \approx 2.6144$

When gets the free string to zero?

1. Parameterize free strings for coil j (Linkage)
2. Structural properties
3. Successive interacting differential equations
4. Inserting end of parameter interval
5. Coefficients of power series
6. Ph. Flajolet: Singularities
7. Pringsheim's Theorem

$v > v_c$ ($s < ew$), $F(Z)$ has convergence radius R (Flajolet)
all coefficients positive, singualrity $Z = R$ (Pringsheim), $F(Z)$ has no real singularities, contradictions, some coefficient is negative.