**Rheinische Friedrich-Wilhelms-Universität Bonn**
**Mathematisch-Naturwissenschaftliche Fakultät**

# Theoretical Aspects of Intruder Search

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

# Chapter 6

# Escape Paths for the Intruder

In this chapter we would like to discuss a reverse situation. An intruder tries to escape from an environment as soon as possible. This work is inspired by the famous question of Bellman (brought up in 1956) who asked for the shortest escape path from an unknown forest.

As before we would like to consider geometric variants as well as more discrete situations. Again, the problem statement can be considered to be a game. The intruder has some abilities and tries to escape from the environment quickly whereas the adversary can manipulate the environment so that the intruder leaves the environment very late. We are looking for apropriate escape strategies for the intruder and consider different performance measures.

## 6.1 Lost in a forest

Assume that a simple region $R$ in the plane is given which boundary is formally defined by a closed Jordan curve $B$ that divides the plane in two simply connected regions. The intruder is located inside $R$ and tries to find the boundary $B$ as soon as possible by a deterministic escape path in the plane. We assume that the intruder has no sight system and only detects the boundary by touching it. The starting position $p$ inside $R$ and the rotation of $R$ is unknown for the intruder but the exact shape of $R$ is known. For example, somebody is located inside a dark forest of known shape and tries to get out of the forest as soon as possible. The deterministic escape path $\Pi$ has to lead out of the region $R$ for any starting position $p \in R$ and any rotation of $R$ around $p$.

The performance measure for the path $\Pi$ is simply its length. There will be a worst case starting position $p$ of $\Pi$ and a worst case rotation of $R$ so that for $p$ the full path length of $\Pi$ is required to hit the boundary $B$. If no such point exists, there will be a better path $\Pi$ of shorter length.

Considering such escape paths has a long tradition as mentioned above. Unfortunately, the optimal escape paths is only known for some very special shapes and totally unknown for general (polygonal) environments. We first discuss some simple convex situations where the diameter is optimal.

Conversely, the problem can also be considered as a covering problem. Consider the class $C_L$ of rectifiable curves in the plane of some lenght $L$. Find an environment $R$ of *small size* that can be rotated and translated so that it covers any curve of $C_L$. In the literature for $L = 1$ such covers are also denoted as *worm covers*. So we are searching for environments that are worm covers and but at least one worm finally touches the boundary from any starting situation.

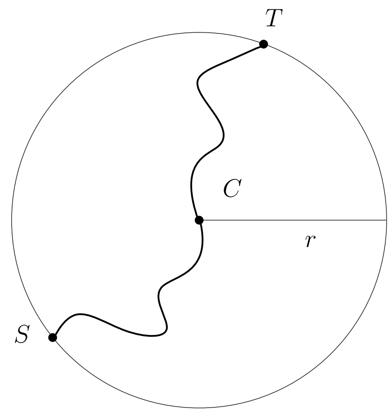


Figure 6.1: A cirlce of radius $r$ covers any path of lenght $< 2r$. The diameter is a shortest escape path.

### 6.1.1 Simple examples and the diameter path

Let us assume that $R$ is a circle of radius $r$.

**Theorem 68** *The shortest escape path from a circle $R$ of radius $r$ is given by the diameter segment of $R$. Conversely, a circle of radius $\frac{1}{2}$ is a worm cover.*

**Proof.** Let us assume that the optimal escaps path $\Pi$ of lenght $L$ is given. We consider the point $C$ on $L$ that splits $\Pi$ into two parts of lenght $L/2$. Let $S$ denote the starting point of $\Pi$ and $T$ denote its endpoint. Now place $C$ to the center of the circle; see Figure 6.1. In order to leave the circle at least $S$ or $T$ has to touch the boundary of $R$, thus $|CT| \geq r$ or $|CS| \geq r$. on the other hand the diameter $d$ of $R$ of lenght $2r$ is an escape path. □

Interestingly, also the semicircle has the same escape path and a semicirlce of diameter 1 is also a worm cover. The proof is a bit more complicated and was given by A. Meir and manifested by Wetzel (1973).

**Theorem 69** *The shortest escape path of a semicircle $R$ of radius $r$ and diameter $2r$ is given by the diameter. Conversely, a the semicircle of radius $\frac{1}{2}$ is a worm cover.*

**Proof.** Let us assume that the escape path is a path with start end endpoint $S$ and $T$. We rotate the path so that $ST$ is in parallel with the base line $B_l$ of the semicircle and we also translate the segment (and the path) so that there is a single tangent point $I$ on the base $B$ and all other points of the path $\Pi$ lie above $B_l$ as depicted in Figure 6.2. Consider an arbitrary point $X \in Pi$, w.l.o.g. we assume that $X$ lies inside the path from $S$ to $I$. We also consider the reflections of $S'$ and $T'$ of $S$ and $T$ along the base line $B_l$. The segment $ST'$ intersects the base line at some point $O$. The length of the segment $SX$ is shorter than $\Pi_S^X$. By refelection the length of the segment $XT'$ is shorter than the path $\Pi_X^T$.

Now we translate the construction so that $O$ is the center of the semicircle. We would like to argue that $|XO| \leq r/2$ holds. This means that any $X$ is inside the semicircle which gives the conclusion.

We consider the triangle $SXT'$ where $O$ divides $ST'$ into two parts of the same length. By geometry we know that the *median* $XO$ is of the triangle is shorter that $\frac{1}{2}$ the length of the adjacent sites.

This means that

$$|XO| \leq \frac{1}{2}(|XS| + |XT'|) \leq \frac{1}{2}(\Pi_S^X + \Pi_X^T) < r\,.$$

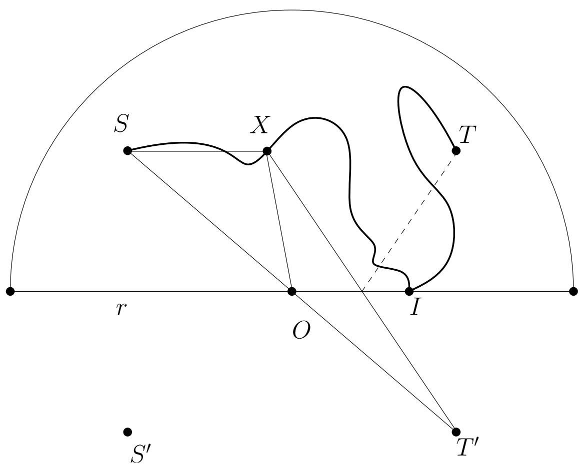


Figure 6.2: A semicirlce of radius $r$ covers any path of lenght $< 2r$. The diameter is a shortest escape path.

$\square$

**Exercise 27** *Show that the* median *of the triangle is always shorter than the average (factor* $\frac{1}{2}$*) of the length of the adjacent sites.*

**Exercise 28** *Show that the above Theorems also hold for closed paths. Note that we considered open paths in the proofs above.*

In general for a given region $R$ the diameter $d$ is defined to be the longest shortest path between two points in $R$. The corresponding points are always located on the boundary of $R$ (otherwise there exist two points connected by a longer shortest path). Thus, any diameter (path) is always an escape path, it need not be an optimal escape path as we will see in the next section. But for some convex objects $R$ the diameter is indeed optimal as we will prove now. Some fatness condition is required.

First we consider the rhombus $R$ of diameter $L$ and angle $\theta = 60\circ$ as depicted in Figure 6.3. We would like to show that any escape path has lenght at least $L$. As already mentioned the diameter of lenght $L$ is an escape path. The following proof stems from Poole and Gerriets (1973).

**Theorem 70** *The optimal escape path for the rhombus $R_\alpha$ of diameter L and angle $\alpha = 60^\circ$ is given by its diameter.*

**Proof.** As in the previous proofs we split an optimal escape path of some lenght $L' < L$ into two halfs of lenght $L'/2$ and consider the mean point $C$. We let $C$ slide along the shorter diagonal $BE$ as shown and rotate the path so that the path $\Pi_S^C$ is tangent to $AB$ and the path $\Pi_C^T$ is tangent to $BD$. Such a rotational center for $C \in BE$ and orientation of $R$ always exists. (Note that in an extreme case $C$ could be located at $B$ and only touches both $AB$ and $BD$.) Let $X \in AB$ and $Y \in BD$ denote the corresponding tangent points of $\Pi_S^C$ and $\Pi_C^T$, respectively.

Because $L'$ is covered by $R$ at least one path $\Pi_S^C$ or $\Pi_C^T$ has to hit the upper angle $AED$ of $R$. W.lo.g. assume that $\Pi_C^T$ hits $AED$. Consider the shortest path from $Y$ to $AED$ and to $BE$ which meet $ED$ at $p_1$ and $BE$ at $p_2$ by angle $\pi/2$, respectively. This means that $\Pi_C^T$ cannot be shorter than the sum of lenghts of $p_1Y$ and $p_2Y$.

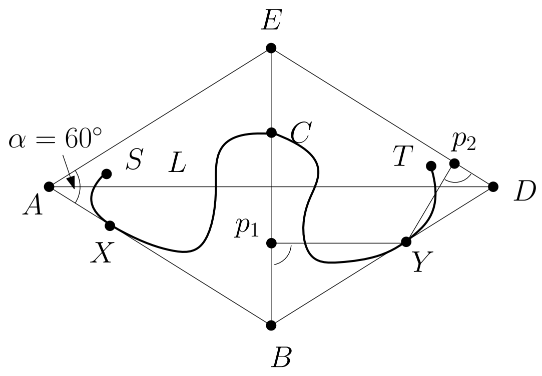


Figure 6.3: The rhombus $R_\alpha$ of diameter $L$ and angle $\alpha = 60°$. The diameter is the shortest escape path since $|p_1Y| + |p_2Y|$ equals $L/2$ for any point $Y$ on $BD$.

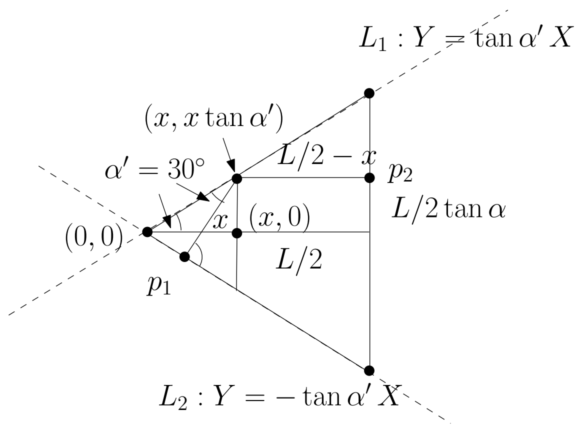


Figure 6.4: Parameterization for the conclusion in Theorem 71. The distance from $(x, x\tan\alpha')$ on $L_1$ to $p_1$ on $L_2$ also equals $x$ for $\alpha' = 30°$.

Finally, we show that $|p_1Y|+|p_2Y|$ equals $L/2$ for all $Y \in BD$, which concludes the proof. This holds by the geometric arguments shown in Figure 6.4. □

**Exercise 29** *Is the shortest escape path always unique? Answer the question for convex and non-convex regions R.*

### 6.1.2 Besicovitch Zig-Zag path

Up to now all shortest escape paths where given by the diameter of the given convex object. This is not always true as we will show by a family of isosceles triangles $T_\alpha$ with base of length

$$b_\alpha = \sqrt{1 + \frac{1}{9\tan^2\alpha}}\,.$$

as shown in Figure 6.5 for $\alpha = 60\circ$ where $T_\alpha$ is a unilateral triangle. We will finally show that we can escape from the unilateral triangle of side-length 1 by a symmetric Zig-Zag path of lenght $\sqrt{\frac{27}{28}} < 1$, although 1 is obviously its diameter.

In general we consider $\alpha$ to be in the interval from roughly 52.24° up to 60°. For $\alpha > 60°$ the base $b_\alpha$ is no longer the largest segment (and not the diameter any more), the reason for $\alpha \geq 52.24°$ is shown below. We show that for any $T_\alpha$ the shortest symmetric Zig-Zag-Path is an escape path for $T_\alpha$ with path lenght smaller than $b_\alpha$. More precisely, the shortest symmetric Zig-Zag path will have lenght 1 and leaves $T_{alpha}$ from any starting point.

The following result goes back to Coulton and Movshovich (2006). First, we define a symmetric Zig-Zag escape path. We orient $T_\alpha$ so that the base $b_\alpha$ runs in parallel to the $X$-axis and runs from $(0,0)$ to $(b_\alpha, 0)$. The remaining segments $l_\alpha$ and $r_\alpha$ of $T_\alpha$ run above the $X$-asis in parallel along the lines $L_1 : Y = \tan\alpha X$ and $L_2 : Y = \tan\alpha(b - X)$, respectively as given in Figure 6.5.

The symmetric Zig-Zag consists of three consecutive segements of the same length and starts starts at the origing of $T_{alpha}$. Any segment has the same altitude $h$ w.r.t. the base $b_\alpha$. The last segment exactly touches the segment $r$ of $T_\alpha$. By construction any such path is an escape path for the corresponding $T_\alpha$.

Now, we would like to construct a Zig-Zag path of length 1 for any $T_\alpha$ such that the path is the shortest symmetric Zig-Zag for a corresponding $b_\alpha$. Such a path is the shortest, if its *straightened path* of the same length hits the line $L : Y = 3\tan\alpha(b_\alpha - X)$ by a right angle as depicted in Figure 6.5 i). By congruence as shown in Figure 6.5 ii) we conclude that $\frac{1}{x} = \frac{b_\alpha}{1}$ which gives $x = \frac{1}{b_\alpha}$. Finally we determine $b_\alpha$ by $y = \tan\alpha\left(b_\alpha - \frac{1}{b_\alpha}\right)$ and $x = \frac{b}{1}$ and $x^3 + (3y)^2 = 1$ which gives

$$b_\alpha = \sqrt{1 + \frac{1}{9\tan^2\alpha}}\,.$$

For $\alpha = 60^\circ$ we have $b_\alpha = \sqrt{\frac{28}{27}}$ and we can escape from the unlateral triangle of side-length 1 by a symmetric Zig-Zag path of lenght $\sqrt{\frac{27}{28}} < 1$, although 1 is obviously its diameter.

For small $\alpha$ there might be other three-segment paths that also have distance 1 or even a shorter distance. This can happen, for example if a line $L_3 : Y = \tan(2\alpha)$ runs in parallel with $L_2$ as shown in Figure 6.6. This means $-3\tan\alpha = \tan 2\alpha$ or $\tan\alpha = \sqrt{\frac{5}{3}}$. To avoid such situations we require $\alpha \geq \alpha_0$ where $\alpha_0$ solves the equation.

**Theorem 71** *For any $\alpha \in [\arctan(\sqrt{\frac{5}{3}}), 60^\circ]$ there is a symmetric Zig-Zag path of lenght 1 that is an escape path of $T_\alpha$ smaller than the diameter $b_\alpha$.*

Before we give a proof for the fact that the corresponding symmetric Zig-Zag escape paths are indeed optimal escape paths for any $T_\alpha$ we first introduce some other models and interpretations of the problem.

## 6.2 Different models and cost measures

Up to now we have considered the case that the intruder tries to escape from a geometric environment without any knowledge of its position. Let us assume that a bit more information is given and let us also consider a somewhat more discrete version. The intruder starts at the source of $m$ long corridors, each of which finally lead out of the environment. The agent also knows the depth $s_i$ but not the correspondance to the corridors.

Consider the situation where a set $L_m$ of $m$ line segments $s_i$ of unknown length $|s_i|$ (which might represent dark corridors) are given and an agent has to find the end of only *one* arbitrary corridor as depicted in Figure 6.7(i). Just choosing a single corridor and move to its end might be very bad, if it is unfortunately the largest corridor. So in this escape problem the agent will move into one corridor $s_{j_1}$ up to a certain distance $x_1$ and then check another corridor $s_{j_2}$ for another distance $x_2$ and so on. Finally he will hit the end of one of the corridors with hopefully overall short path length.

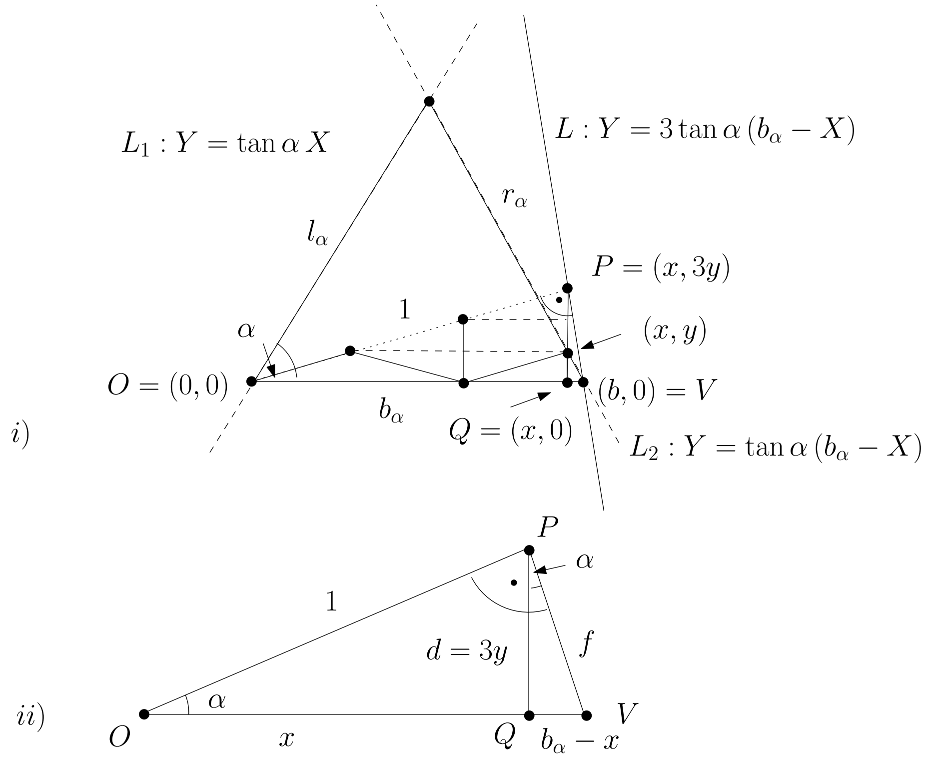


Figure 6.5: i) Any symmetric Zig-Zag path for $T_\alpha$ is an escape path. For the shortest such path the straightened segment mets the line $L : Y = 3\tan\alpha(b_\alpha - X)$ by a right angle. ii) Using the congruent triangles $OPQ$ and $OPV$ we have $\frac{1}{x} = \frac{b}{1}$.

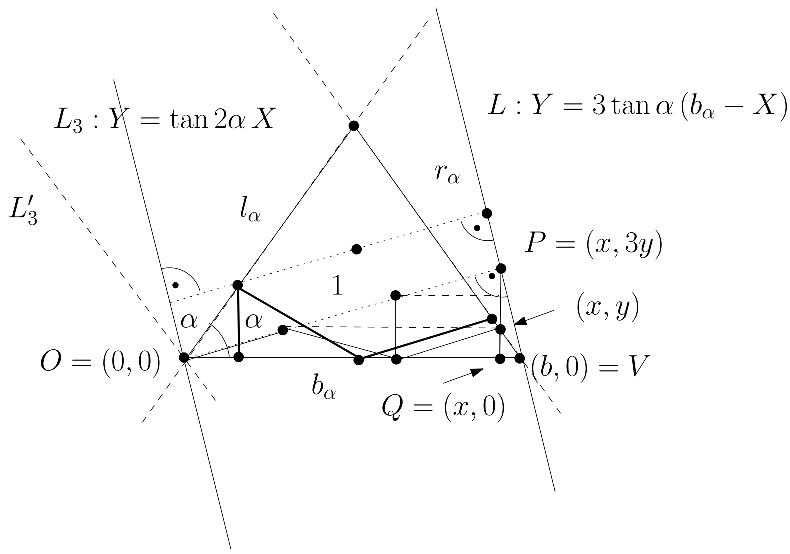


Figure 6.6: If $\alpha$ is too small, other Zig-Zag path might have a better performance (lenght $\leq 1$).

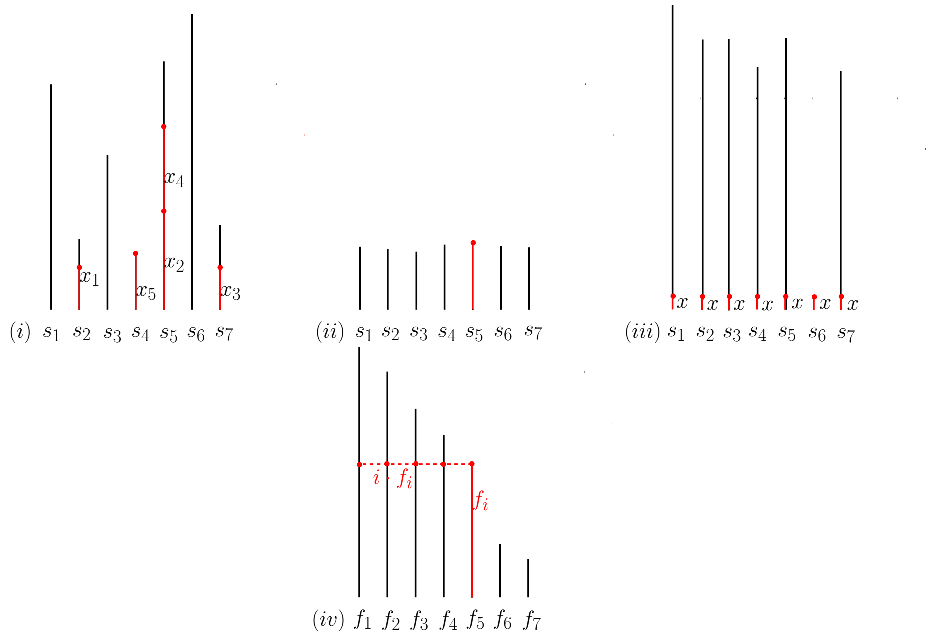


Figure 6.7: (i) Online searching for the end of one segment (or digging for oil) for $m = 7$ segments of unknown length as considered by Kirkpatrick. A reasonable strategy changes successively from one segment to the other with distances $x_i$ on $s_{j_i}$. It is allowed to resume searching at the previous end-point with no extra cost (see $x_2$ and $x_4$ on $s_{j_2} = s_{j_4} = s_5$). The strategy reaches the end of segment $s_5$ after 5 movements $(x_1, \ldots, x_5)$. If at least the distance distribution is known, the problem is easier to solve. There are two extreme cases. (ii) If all segments have almost have the same length, it is reasonable to move along an arbitrary segment with largest distance, this is almost optimal. (iii) If there is one segment of very short length and all other segments are very long, one will find the end of the short segment by checking all segments with the shortest distance. This path is also short. (iv) In general the *discrete certificate* is defined for the given distance distribution. If $f_1 \geq f_2 \geq \cdots \geq f_m$ is the order of the length of the segments, it is always sufficient to check $i$ arbitrary segments with length $f_i$ and $\min_i i \cdot f_i$ is the best such strategy.

Kirkpatrick (2009) introduced this problem and also motivates the situation by the scenario of digging for oil at $m$ locations $s_i$ where the distance $|s_i|$ to the source of the oil of place $s_i$ is not known. It is sufficient to get to the source of one place and the overall effort should be small. In this scenario it is allowed to resume the movement (or digging) for a location $s_i$ at the endpoint where $s_i$ was left at the previous visit; see Figure 6.7(i). There are no extra costs for moving to the previously reached depth at $s_i$.

Now we define a performance measure. Let us assume that all $m$ distances $|s_i|$ are known but not the correspondence to the places $s_i$. In this case we can sort the $m$ distances and obtain a discrete distance distribution of the length of the segments.

First, consider the extreme situations in Figure 6.7(ii) and (iii) If all segments $s_i$ almost have the same length, a successful strategy will simply use the maximal length $x$ among all segments, checks this for an arbitrary segment and will succeed with path length $x$ in the worst case; Figure 6.7(ii). If on the other extreme (see Figure 6.7(iii)) the distance to a single source is small but the distances are very large to all others (and we only know this distribution), the best option is to check all segments successively by the small distance $x$. This gives an effort of at most $x \cdot m$ in the worst case when the small segment is found at the latest visit.

In general Kirkpartrick defines a *certificate* that takes the distance distribution into account.

The segments are sorted by distance $s_i$. Let $f_j$ with $j = 1, \ldots, m$ denote the sorted list of the lengths of the segments with decreasing distances $f_j$ as shown in Figure 6.7(iv). Now for distance $f_i$ there are exact $i$ segments that are larger than or equal to $f_i$. Thus, after checking arbitrary $i$ segments $s_j$ with distance $f_i$ it is clear that we will find the end of at least one segment. In the worst case the $i$ largest segments have been checked including $f_i$. This is the discrete certificate $\Pi(f_i)$ for distance $f_i$ of length $i \cdot f_i$. The overall certificate is given by

$$\min_{1 \le i \le m} i \cdot f_i =: \mathrm{cert}(L_m) \ .$$

So let us assume that in the online sense the distances are totally unknown to the agent. Kirkpatrick defines a general strategy that always approximates the certificate whithin a factor of $O(\mathrm{cert}(L_m) \log(\min(m, \mathrm{cert}(L_m)))$ for any list $L_m$ of $m$ totally unknown segments. It is shown that this factor is tight. The corresponding *dovetailling* strategy subdivides the overall digging length successively in a logarithmic way among an arbitrary order of the segments.