Theoretical Aspects of Intruder Search

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail elmar.langetepe@informatik.uni-bonn.de. Thanks in advance!
Chapter 3

Discrete Cop and Robber game

In this chapter we would like to discuss another discrete variant of the intruder search problem. In comparison to the previous chapter, we assume that at any time of the game the position of the single intruder is given.

More precisely, there is a single robber $R$ and a set of cops $C$ and a graph $G = (V,E)$. The game starts with the cops, by choosing the starting vertices for the set $C$. After that, the robber $R$ can choose its starting vertex. The game runs in subsequent steps. First, any cop can move from a vertex to an incident vertex, then the robber can move. The game ends, when one cop enters the position of the robber or the robber enters the position of a cop, respectively.

Cop and Robber game for graphs:

Instance: A Graph $G = (V,E)$ and the cardinality of the cops $C$.

Question: Is there a winning strategy $S$ for the cops $C$?

We are searching for classifications of graphs that allow a winning strategy for $C$ or vice versa a winning strategy for $R$. Aigner and Fromme introduced the problem in the midst of the 90ies.

3.1 Classifications of graphs

3.1.1 Simple examples and pitfalls

It is interesting to see that it makes a difference, if we do not allow the robber to keep in place during its strategy. This is called the active version of the game, in correspondence to the passive version, where the robber is not forced to move in any step.

Figure 3.1 shows an example where this makes a difference for a single cop. In the active version the cop starts at vertex $v_1$ and the robber can only choose the opposite vertex $r_2$. The cop moves toward $v$. Now the robber has to move to $r_1$. The cop moves toward $v_2$ and after the next mandatory move of the robber, the robber will be caught. In the passive version the robber can move around or rest in the 4-cycle and holds distance 2 from the cop all the time.

In the following we will always discuss the more intuitive passive version of the game. Let $G_C$ denote the set of all graphs that allow a winning strategy for $C$ and let $G_R$ denote all graphs that have a winning strategy for $R$.

Obviously, any tree $T$ belongs to $G_C$ already for a single cop, that successively moves into the subtree of $R$. Additionally, for a single cop, all graphs that contain a cycle of length at least 4 belong to $G_R$.

We concentrate on a single cop. In the winning case for the cop, the final situation is as follows: The robber is located in a vertex $v_r$ and the cop is located in $v_c$ for an edge $e = (v_r,v_c)$. Moreover, all neighbors, $N(v_r)$, of $v_r$ are also neighbors of $v_c$, which means $N(v_r) \subseteq N(v_c)$.
For a pair \((v_r, v_c)\) of vertices we call \(v_r\) a pitfall and \(v_c\) its dominating vertex if \(N(v_r) \cup \{v_r\} \subseteq N(v_c)\) holds. Obviously, a graph \(G\) without a pitfall is in \(G_R\). Figure 3.2 shows an example.

**Exercise 16** Present a construction scheme for graphs of arbitrary size without pitfalls.

### 3.1.2 Algorithmic approaches

We would like to show that for a single cop the classification of a graph depends on the successive removement of pitfalls of \(G\).

**Lemma 34** Let \(v_r\) be a pitfall of some graph \(G\). Then

\[ G \in G_C \iff G \setminus \{v_r\} \in G_C, \]

where \(G \setminus \{v_r\}\) results from \(G\) by removing all edges adjacent to \(v_r\) and vertex \(v_r\) from \(G\).

**Proof.** If \(G \setminus \{v_r\} \in G_R\) holds, the robber simply identifies any visit of the cop of the pitfall \(v_r\) by the dominating vertex \(v_c\) and makes use of a strategy in \(G \in G_R \setminus \{v_r\}\). Note that the neighborship of the pitfall \(v_r\) is a subset of the neighborship of \(v_c\). This means that we do not weaken the cop by the identification.

If \(G \setminus \{v_r\} \in G_C\) holds, the cop wants to extend its winning strategy to \(G\). The cop simply identifies any visit of the robber of the pitfall \(v_r\) as a visit of the dominating vertex \(v_c\) and makes use of the same strategy. Again, since the neighborship of the pitfall \(v_r\) is a subset of the neighborship of \(v_c\), we do not weaken the robber by this identification. If at the end we catch the
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imaginary robber at $v_c$ and the robber is physically located at $v_r$, we will catch the robber in the next step. □

Now, we have a simple characterization of $G_C$.

**Theorem 35** The graph $G$ is in $G_C$, if and only if the successive removement of pitfalls finally ends in a single vertex. The classification of a graph can be computed in polynomial time.

**Proof.** Lemma 34 gives the key argument, as the classification does not change by removing pitfalls. This means that we either end up in a graph with no pitfalls for $G \in G_R$ or in a single vertex for $G \in G_C$.

Checking the existence of a pitfall can be done locally for any vertex and its neighborhood. After computing the neighborhood sets, we can check the pitfall property for a vertex in a brute-force manner in $O(n^2)$ time and for all vertices in $O(n^3)$ time for a graph with $n$ vertices. At most $n$ reduction steps can be done. □

**Exercise 17** Design an efficient algorithm for checking the pitfall property of a single vertex and/or for the graph.

The above shrinking process answers the classification question algorithmically in polynomial time. On the other hand we would like to construct arbitrary examples of representatives of $G_C$.

It can be shown that $G_C$ is closed under the operations product of two graphs and reduction of a graph.

The product $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined by vertex set $V_1 \times V_2$ and an edge set by the following rules: $(v_1, v_2)$ and $(w_1, w_2)$ of $V_1 \times V_2$ build an edge if:

1. $v_1 = w_1$ and $(v_2, w_2) \in E_2$ or
2. $(v_1, w_1) \in E_1$ and $v_2 = w_2$ or
3. $(v_1, w_1) \in E_1$ and $(v_2, w_2) \in E_2$.

**Lemma 36** If $G_1, G_2 \in G_C$, then $G_1 \times G_2 \in G_C$

**Proof.** If the cop has a winning strategy for $G_1$ that starts in $v_1^s$ and catches the robber in $v_1^r$ and for $G_2$ that starts in $v_2^s$ and catches the robber in $v_2^r$, the cop can start in $(v_1^s, v_2^s)$ apply the strategies simultaneously and finally catches the robber in a vertex $(v_1^r, v_2^r)$. This strategy is obviously correct. □

For a graph $G$ and its subgraph $H$, the retraction from $G$ to $H$ is a mapping $\varphi : V(G) \rightarrow V(H)$ of the vertices of $V(G)$ of $G$ to the vertices $V(H)$ of $H$ as follows: $\varphi(h) = h$ for $(u, v) \in E$ we have $(\varphi(v), \varphi(u)) \in E(H)$. The graph $H$ is called a retract of $G$, if a retraction from $G$ to $H$ exists.

Note that $G \setminus \{v_r\}$ for a pitfall $v_r$ is a retract of $G$.

**Lemma 37** If $G \in G_C$, and graph $H$ is a retract of $G$, then $H \in G_C$.

**Proof.** Assume that $H \in G_R$ holds and let $\varphi$ be the mapping for a retraction from $G$ to $H$. We would like to show $G \in G_R$. We extend the winning strategy for $H$ to a winning strategy of $G$ as follows: $R$ remains in $H$ and identifies the moves of $\bar{C}$ in $G$ as moves in $H$. That is, if $C$ moves from $v$ to $u$ in $G$, the robber identifies this move as a move from $\varphi(u)$ to $\varphi(v)$ which
exists in $H$ by definition of $\varphi$. The robber always moves according to the strategy in $H$ and cannot be caught.

Note that, the above lemmata do not rely on the fact that there is only one cop.

**Theorem 38** The class of graphs $G$ in $G_C$ is closed under the operations product and retraction.

### 3.1.3 How many cops are required?

Obviously, any graph with a 4-cycle will not belong to $G_C$, therefore it makes sense to think about more than one cop. For a graph $G$ the cop-number, $c(G)$, denotes the minimum number of cops required to guarantee that $G \subseteq G_C$ holds.

A **vertex cover** of a graph $G$ is a subset $V_c \subseteq V$ so that any vertex $u \in V \setminus V_c$ has a neighbor in $V_c$. Therefore the minimum vertex cover is an upper bound on $c(G)$. First, we show that $c(G)$ can be arbitrarily large for some graphs.

**Theorem 39** Let $G = (V, E)$ be a graph with minimum degree $n$ that contains neither 3- nor 4-cycles. We conclude $c(G) \geq n$.

**Proof.** Let us assume that $n - 1$ cops are sufficient. If $G$ does not have a vertex cover of size smaller than $n$, the $n - 1$ cops located in the beginning at $c_1, \ldots, c_{n - 1}$ cannot prevent the robber to choose a safe vertex. So the robber chooses such a vertex whose neighbors are not occupied by the cops. Since there are no 3- and 4-cycles, by the next move a single cop cannot threaten (occupy and/or be adjacent to) two neighbors of the robber in the next step. Therefore, there is still one safe neighbor for the robber after the next move of the cops.

It remains to show that a vertex cover of size $< n$ does not exist. Consider any vertex set $V = \{v_1, \ldots, v_{n - 1}\}$ of $G$ and a vertex $w \neq v_i$ for $i = 1, \ldots, n - 1$. Note that $|V| \geq n$ holds, so $w$ exists. Now consider the neighborhood, $N(w)$, of $w$. Let it consist of $k$ vertices $v_1, \ldots, v_k$ from $V$ and $l - k$ vertices $w_1, \ldots, w_{l - k}$ not in $V$. We have $l \geq n, k \leq n - 1$ and $l - k \geq 1$. There are no 3- and 4-cycles, so $N(w_i) \cap N(w_j)$ has to be $\{w\}$ for $i \neq j$. If the set $V$ is a vertex cover for $G$, any $N(w_i)$ has to contain a different vertex from $V$. But none of the $N(w_i)$'s can contain a vertex of $v_1, \ldots, v_k$, since this would give a 3-cycle with $w$. This means that we require $l - k$ different vertices from $v_{k + 1}, \ldots, v_{n - 1}$ and $n$ vertices from $V$ in total, a contradiction. □

We can construct regular graphs of arbitrary size, which fulfill the condition of Theorem 39. The following Theorem is given by construction.

**Theorem 40** For every $n$ there exists a graph without 3- or 4-cycles with minimum degree $n$. So, for any $n$ there is a graph with $c(G) \geq n$.

**Proof.** For $n = 2$ the simple 5-cycle will work. Note that $C_5$ is 3-colorable, which means that we color the vertices by three colorsa such that no two colors are adjacent. Three colors are required and sufficient for $C_5$. Inductively, we construct a 3-colorable graph with degree exactly $n$ for any vertex $v$ of $G$ and without 3- and 4-cycles. The coloring is required for maintaining the cycle condition by construction.

The induction base for $n = 2$ was shown above. Let us assume that the statement holds for $n$. We consider four copies $G_0, G_1, G_2, G_3$ of a corresponding graph $G$ for $n$ as depicted in Figure 3.3. We build new edges with respect to the coloring of the vertices so that any vertex obtains an additional edge; see Figure 3.3. From $G_i$ to $G_j$ the exact copies of two vertices of a single color are connected. For example from $G_1$ to $G_2$ all identical copies of color 3 are
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Connected, from \( G_2 \) to \( G_0 \) all identical copies of color 2 are connected and so on. There is a unique correspondence as shown in Figure 3.3.

Since there are no cycles of size 3 or 4 in \( G_0, G_1, G_2, G_3 \) and any two edges between \( G_i \) and \( G_j \) make use of identical copies of the same color there are at least two edges between them in \( G_i \) and \( G_j \) respectively. So there are no 3- or 4-cycles in the new graph. For the inductive step, we require a new 3-coloring, which will be attained by interchanging the colors for example color 3 by 1 and color 1 by 3 in \( G_1 \) and color 2 by 1 and color 1 by 2 in \( G_2 \) and so on. Thus, we maintain 3-coloring in \( G_0, G_1, G_3, G_4 \) and also for the connections.

Finally, in this section we prove some positive results by bounding the cop-number from above for special graphs. The corresponding proofs are constructive, i.e., a winning strategy for the cops can be computed.

**Theorem 41** Consider a graph \( G \) with maximum degree 3 and the property that any two adjacent edges are contained in a cycle of length at most 5. Then \( c(G) \leq 3 \).

**Proof.** The proof is constructive in the following sense. If the position of the robber is known, for the cops \( c_1, c_2 \) and \( c_3 \) we consider three paths towards \( r \) that use all incident edges to \( r \). We choose \( P_1, P_2 \) and \( P_3 \) for \( c_1, c_2 \) and \( c_3 \) respectively. The paths cover the incident edges by different cops and with length \( l_1, l_2 \) and \( l_3 \). And the paths make use of any possible shortcut for reaching the incident edges. Note that the paths need not be disjoint and \( r \) might also have only one or two incident vertices. But such paths do always exist. We would like to argue that by the condition of the Theorem, we can decrease the overall distance \( l := l_1 + l_2 + l_3 \) in any move of the cops.

Formally, after the move of the robber, \( R \), we move \( c_1, c_2 \) and \( c_3 \) to \( c'_1, c'_2 \) and \( c'_3 \) so that \( l' < l \) holds. We further assume that \( r \) was adjacent to exactly three vertices \( r_1, r_2 \) and \( r_3 \). The other cases can be handled analogously, and are given as an exercise. We have \( P_1 = \{c_1, \ldots, r_1, r\} \), \( P_2 = \{c_2, \ldots, r_2, r\} \) and \( P_3 = \{c_3, \ldots, r_3, r\} \) and consider the following cases.

1. The robber \( R \) stands still. The cops move along the paths toward \( R \) and \( l' \leq l - 3 \).
2. The robber $R$ moves to $r_1$ (w.l.o.g.):

- **$r_1$ has degree 1**: This cannot happen since $(r,r_1)$ and $(r,r_2)$ are neighboring edges.
- **$r_1$ has degree 2**: Either $c_1$ was on $r_1$ and we are done or move all three cops toward $r$ which gives $l' \leq l_1 - 2 + l_2 + l_3 = l - 2 < l$.
- **$r_1$ has degree 3**: Either $c_1$ was on $r_1$ and we are done or we have $l_1 \geq 2$. At least one adjacent vertex, say $x$, of $r_1$ does not belong to $P_1$, otherwise we use a shortcut for $P_1$. This means that $(x,r_1)$ and $(r_1,r)$ are on a cycle of length at most 5. Since $r$ has only degree 3, one of the vertices $r_2$ or $r_3$, (say $r_2$) also belong to this cycle as depicted in Figure 3.4. So we have a 5-cycle $r_2, y, x, r_1, r$. We move all three cops toward $r$, respectively $r_1$ and use the paths $P_1 = \{c'_1, \ldots, r_1\}$, $P_2 = \{c'_2, \ldots, r_2, y, x, r_1\}$ and $P_3 = \{c'_3, \ldots, r_3, r, r_1\}$ with length $l' \leq l_1 - 2 + l_2 + 1 + l_3 = l - 1 < l'$.

In any case we can reduce the distance of the cops to the robber for the paths to all all three neighboring vertices.

Finally, we would like to prove that for any planar graph $G$, indeed $c(G) \leq 3$ holds. We first show that in any graph $G$ it is always possible to protect a shortest path between two vertices by two cops. Protection means the robber cannot enter the path without being caught in the next step. The path length is given by the number of edges along a path between to vertices in $G$. By this measure the triangle inequality holds.

**Lemma 42** Consider a graph $G$ and a shortest path $P = s, v_1, v_2, \ldots, v_n, t$ between two vertices $s$ and $t$ in $G$, assume that we have two cops. After a finite number of moves the path is protected by the cops so that after a visit of the robber $R$ of a vertex of $P$ the robber will be caught.

**Proof.** First, we move a cop $c$ onto some vertex $c = v_i$ of $P$. Let $d(x,y)$ denote the distance between two vertices in $G$. The robber $R$ located at $r$ can only have a shorter distance to vertices on one side of $P$ w.r.t. $c$ because the triangle inequality holds. Assuming, that $r$ is closer to some $x$ in $s, v_1, \ldots, v_i$ and some $y$ in $v_{i+1}, \ldots, v_n, t$ is a contradiction to the shortest path between $x$ and $y$. That is $d(x,c) + d(y,c) \leq d(x,r) + d(r,y)$. This means that $d(r,x) < d(c,x)$ only holds at most for one side of $P$ w.r.t. $c$ and for the other side we conclude $d(r,y) > d(c,y)$ in this case.
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Thus, we move $c$ towards the vertices $x$. Now the robber can move. Again, if there are still vertices on one side of $P$ w.r.t. $c$ which are closer to $r$ than to $c$ we move further on toward these vertices. So finally, we achieved

$$d(r, v) \geq d(c, v) \text{ for all } v \in P$$

by this process.

Now the robber again could make its move. We show that we can maintain the inequality at all time, which means that the robber will be caught if it tries to move towards the vertices of $P$.

Assume Equation 3.1 holds. The robber can either stay in its place, so the cop $c$ does and we fulfill Equation 3.1 (and the second cop could move now). Or the robber moves and tries to contradict Equation 3.1 by its single move. Assume $r$ goes to some vertex $r'$, we have

$$d(r', v) \geq d(r, v) - 1 \geq d(c, v) - 1 \text{ for all } v \in P.$$ 

If there is again some $v' \in P$ with $d(c, v') - 1 = d(r', v')$, we have the same situation as above and we can move $c$ toward $v'$ and Equation 3.1 holds again. Again as before the movement toward $r'$ cannot reduce the distance to $x$ and $y$ on opposite sides of $c$ w.r.t. $P$. Thus, by the move toward some $v'$ we fulfill Equation 3.1.

Finally, we exploit the above property for planar graphs and by the use of 3 cops and two such paths.

**Theorem 43** For any planar graph $G$ we have $c(G) \leq 3$.

**Proof.** We show that the region $R_i$ for the robber $R$ will shrink successively, that is $R_{i+1} \subset R_i$ after some moves of the cops. Two situations can appear.

**Case 1:** All three cops occupy a single vertex $c$ and the robber is located in one component $R_i$ of $G \setminus \{c\}$; see Figure 3.5.

**Case 2:** There are two different paths $P_1$ and $P_2$ from $v_1$ to $v_2$ that are protected in the sense of Lemma 42 by cops $c_1$ and $c_2$; see Figure 3.6. In this case $P_1 \cup P_2$ subdivides $G$ into an interior, $I$, and an exterior region $E$. That is $G \setminus (P_1 \cup P_2)$ has at least two components. W.l.o.g. we assume that $R$ is located in the exterior $E = R_i$.

We will show that these two cases can appear successively and the region $R_i$ of the robber will shrink. In the beginning all cops are located in a single vertex $c$ and we are in case 1. We show how we handle the cases.
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Figure 3.6: Case 2: Two cops protect two paths.

**Movements in Case 1:** We consider different situations for the neighbors of $c$:

- **$c$ has one neighbor in $R_i$:** Move all cops to this neighbor $c'$ and consider $R_{i+1} = R_i \setminus \{c'\}$. This gives Case 1 again.

- **$c$ has more than one neighbor in $R_i$:** Let $a$ and $b$ be two of the neighbors and let $R(a, b)$ be a shortest path in $R_i$ between $a$ and $b$. One cop remains in $c$, another cop protects the path $P(a, b)$ by Lemma 42. Thus $P_1 = a, c, b$ and $P_2 = P(a, b)$. We are in Case 2 with $R_{i+1} \subset R_i$.

**Movements for Case 2:** We consider the situation for the composition of $R_i$ and the location of the robber. We first assume that there is another shortest path $P'(v_1, v_2)$ but different from $P_1$ and $P_2$ that partially runs in $R_i$ and also connects $v_1$ and $v_2$.

Let $x_1$ be the first vertex of $P'(v_1, v_2)$ where the path $P'$ leaves $P_1 \cup P_2$ after $v_1$ and $x_2$ be the first vertex of $P'$ after $x_1$ that enters $P_1 \cup P_2$ again.

We let $c_3$ protect the path $P_3$ which results from combining $P_{1,2}(v_1, x_1)$ with $x_1, r_1, \ldots, r_l, x_2$ and with $P_{1,2}(x_2, v_2)$ as depicted in Figure 3.7. While $c_1$ and $c_2$ protect $P_1$ and $P_2$, the cop $c_3$ can protect this path $P_3$ from $v_1$ to $v_2$. At the end $c_3$ protects $P_3$ and $c_1$ or $c_2$ protect the remaining path, we are in Case 2 with $R_{i+1} \subset R_i$.

On the other hand, if there is no path different from $P_1$ and $P_2$ and partially running in $R_i$ that connects $v_1$ and $v_2$, there are no such leave and entry vertices $x_1$ and $x_2$ that are connected in $R_i$. Thus, the robber has to be inside a component that fully is connected to a single vertex $x$ on $P_1$ and $P_2$. Thus, we move $c_3$ to this vertex, and also $c_1$ and $c_2$ and end in Case 1 again. □
Figure 3.7: The situation for the two chains $P_1$ and $P_2$ protected by $c_1$ and $c_2$. If there is another shortest path from $v_1$ to $v_2$ different from $P_1$ and $P_2$ that runs partially in $R_i$, we construct a path $P_3$ that runs from $v_1$ to $v_2$ that is protected by $c_3$ alone and protects vertices of $R_i$. If there is no such path, a vertex $x$ exists that can be visited by all cops and gives Case 1 again.