Theoretical Aspects of Intruder Search
Course Wintersemester 2015/16
Geometric Firefighting Plane

Elmar Langetepe

University of Bonn

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Expanding fire in the plane
• Barrier curve with speed \( v > 1 \)
• Current point outside the fire

*Geometric Firefighter Problem in the plane*

**Instance:** Expanding fire-circle spreads with unit speed from a given starting point \( s \), start radius \( A \).

**Question:** How fast must a firefighter be, to build a barrier that finally fully encloses and stops the expanding fire?
Spiral movements for speed $v$

Start on the boundary, speed $v > 1$

Allowed angle?

Riding the fire Log. Spiral around $Z$

Excentricity $\alpha \cos(\alpha) = 1$

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Spiral movements for speed $v$

- Start on the boundary, speed $v > 1$
Spiral movements for speed $v$

- Start on the boundary, speed $v > 1$
- Allowed angle?
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Spiral movements for speed $v$

- Start on the boundary, speed $v > 1$
- Allowed angle?
- *Riding* the fire

\[
\alpha = \arccos \left( \frac{1}{v} \right)
\]

\[
x \cos(\alpha)
\]
Spiral movements for speed $v$

- Start on the boundary, speed $v > 1$
- Allowed angle?
- *Riding* the fire
- Log. Spiral around $Z$
- Excentricity $\alpha$
  \[ \cos(\alpha) = \frac{1}{v} \]
Properties of a spiral

- Polar coordinates $S(\varphi) := (\varphi, a \cdot e^{\varphi \cot \alpha})$
- Constant $a$
- $\alpha \in (0, \pi/2)$, $\cot \alpha$ from 0 to $\infty$
- $|S_q^p| = \frac{1}{\cos \alpha} (|Bq| - |Bp|)$
Spiralling strategy, upper bound on the speed

Bressan et al. 2008

- Spiral was constructed
- Let the fire expand
- Follows to current point $D$
- Speed difference?
- $\gamma(\beta) =: \gamma$
- $\overline{OC} = a$

\[
a \cdot e^{(2\pi + \gamma) \cot \beta} = \frac{a}{\sin(\beta - \gamma)} \iff e^{(2\pi + \gamma) \cot \beta} = \frac{\sin \beta}{\sin(\beta - \gamma)}.
\]
Spiralling strategy, upper bound on the speed

\[ f(\beta) := \frac{|\text{Length spiral from } O \text{ to } D|}{|\text{Length spiral from } O \text{ to } C| + |\text{Length segment } CD|} \cdot \]

\[ f(\beta) = \frac{\frac{1}{\cos \beta} e^{(2\pi + \gamma) \cot \beta}}{\frac{1}{\cos \beta} + \frac{\sin \gamma}{\sin \beta} e^{(2\pi + \gamma) \cot \beta}} = \frac{\frac{1}{\cos \beta} \sin \beta}{\frac{1}{\cos \beta} + \frac{\sin \gamma \sin \beta}{\sin \beta \sin(\beta - \gamma)}} = \frac{1}{\cos \gamma} \cdot \]

- \[ CD = a \frac{\sin \gamma}{\sin \beta} e^{(2\pi + \gamma) \cot \beta} \]
- \[ \frac{a}{\cos \beta} e^{(2\pi + \gamma) \cot \beta} \]
- \[ \frac{a}{\cos \beta} \]

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Properties of $f(\beta)$

$$f(\beta) = \frac{1}{\cos \beta} \cdot \frac{\cos \beta e^{(2\pi + \gamma) \cot \beta}}{\sin \gamma e^{(2\pi + \gamma) \cot \beta}} = \frac{1}{\cos \gamma}.$$ 

$$\lim_{\beta \to 0^+} f(\beta) = \lim_{\gamma \to 0^-} \frac{1}{\cos \gamma} = 1.$$ 

$$\limsup_{\beta \to \pi/2^+} f(\beta) \leq \limsup_{\beta \to \pi/2^+} e^{(2\pi + \gamma) \cot \beta} \leq \lim_{\beta \to \pi/2^+} e^{(5\pi/2) \cot \beta} = 1.$$ 

Graph showing $f(\beta)$ with key points and asymptotic behavior.
Properties of a spiral

- Maps to 1 at the boundary
- $\gamma(\beta)$ continuous, well-defined, $f$ continuous, well-defined
- Unique global maximum: $v_l := \max_{\beta \in (0, \pi/2)} f(\beta)$.
- Numerically: $\beta_l = 1.29783410242\ldots$ and gives $v_l = f(\beta_l) = 2.614430844\ldots$ and $\gamma(\beta_l) = 1.178303978\ldots$
Construct strategy with speed $\nu > \nu_l = 2.614430844\ldots$

- For any speed $\nu > \nu_l$ spiral keeps in front of fire
- Use $\nu_1 = \frac{1}{\cos \beta_1} > \nu_l$ and spiral with excentricity $\beta_1$
Construct strategy with speed $v > v_l = 2.614430844 \ldots$

- Use $v_1 = \frac{1}{\cos \beta_1} > v_l$ and spiral with excentricity $\beta_1$
- Make it a legal start spiral: $\beta_1$ helps for starting!!
- Starting circle construction:

  - $v' = \frac{1}{\cos \beta'}$ with $v_1 > v' > v_l$
  - $F$ is met $t_1 = t \cos \beta_1$,
    $t_2 = t \cos \beta' > t_1$
  - $x = t(\cos \beta' - \cos \beta_1)$ time from $N$ to $D$ for the fire
Construct strategy with speed $v > v_l = 2.614430844 \ldots$

- $x = t(\cos \beta' - \cos \beta_1)$ time from $N$ to $D$
- Use $x$ for the start

- $\frac{1}{\cos \beta'}(\epsilon_1 + \epsilon_2) - \epsilon_2 < x$
- Speed $v_1 = \frac{1}{\cos \beta_1} > v' > v_l$ helps
- Angle $\beta_1$ helps!
Construct strategy with speed \( v > v_l = 2.614430844 \ldots \)

- For any \( v_1 = \frac{1}{\cos \beta_1} > v_l \)
- Admissible spiral, starting radius \( C_1 = (\epsilon_1 + \epsilon_2) \), excentricity \( \beta_1 \)
2. Enclosure by iterations

- $F = (e^{2k_1 \pi \cot \beta_1}, 0)$ with excentricity $\beta_2 > \beta_1$
  and starting radius $C_2 = C_1 e^{2k_1 \pi \cot \beta_1}$
- Admissable, if $\beta_2 > \beta_1$ close to $\beta_1$. 
2. Iteration

- Spiral with $\beta_2$ until angle $2k_2\pi$
- $F_2$ and the fire is behind at $D_2$
- $\beta_1 < \beta_2 < \ldots < \beta_l$ and spirals $C_i e^{2\pi k_i \cot \beta_i}$
2. Many iterations $\beta_m > \beta_{m-1} > \cdots > \beta_1$

\[
\frac{1}{\cos \beta_m} > \frac{1}{\cos \beta_1} \left( \frac{1}{\cos \beta_m} e^{2\pi \cot \beta_m} + (e^{2\pi \cot \beta_m} - 1) \right)
\]

- $\frac{1}{\cos \beta_m} e^{2\pi \cot \beta_m}$ and $e^{2\pi \cot \beta_m} - 1$
- Versus $\frac{1}{\cos \beta_m}$ (scaling!)
2. Many iterations $\beta_m > \beta_{m-1} > \cdots > \beta_1$

\[
\frac{1}{\cos \beta_m} > \frac{1}{\cos \beta_1} \left( \frac{1}{\cos \beta_m} e^{2\pi \cot \beta_m} + (e^{2\pi \cot \beta_m} - 1) \right)
\]

- Example. $\beta_1 \approx 1.191388 \ldots$ and $\frac{1}{\cos \beta_1} = 2.7$ we require $\beta_m > 1.4268$. 

![Diagram with speed and angles]
2. Many iterations $\beta_m > \beta_{m-1} > \cdots > \beta_1$

**Theorem 56:** (Bressan et al. 2008) For any speed $v > v_l \approx 2.614430844$ there is a spiralling strategy that finally encloses an expanding circle that expands with unit speed.