Theoretical Aspects of Intruder Search
Course Wintersemester 2015/16
Cop and Robber Game Cont./Randomizations

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Theorem 40: For any planar graph $G$ we have $c(G) \leq 3$.

Proof:

- Two cops protect some paths, the third cop can proceed!
**Lemma 39:** Consider a graph $G$ and a shortest path $P = s, v_1, v_2, \ldots, v_n, t$ between two vertices $s$ and $t$ in $G$, assume that we have two cops. After a finite number of moves the path is protected by the cops so that after a visit of the robber $R$ of a vertex of $P$ the robber will be caught.

- Move cop $c$ onto some vertex $c = v_i$ of $P$
- Assuming, $r$ closer to some $x$ in $s, v_1, \ldots, v_{i-1}$ and some $y$ in $v_{i+1}, \ldots, v_n, t$. Contradiction shortest path from $x$ and $y$
- $d(x, c) + d(y, c) \leq d(x, r) + d(r, y)$
- Move toward $x$, finally: $d(r, v) \geq d(c, v)$ for all $v \in P$
- Now robot moves, but we can repair all the time
- $r$ goes to some vertex $r'$ and we have $d(r', v) \geq d(r, v) - 1 \geq d(c, v) - 1$ for all $v \in P$.
- Some $v' \in P$ with $d(c, v') - 1 = d(r', v')$ exists, move to $v'$
**Theorem 40:** For any planar graph $G$ we have $c(G) \leq 3$.

**Proof:**

Case 1: All three cops occupy a single vertex $c$ and the robber is located in one component $R_i$ of $G \setminus \{c\}$.

Case 2: There are two different paths $P_1$ and $P_2$ from $v_1$ to $v_2$ that are protected in the sense of Lemma 39 by cops $c_1$ and $c_2$. In this case $P_1 \cup P_2$ subdivided $G$ into an interior, $I$, and an exterior region $E$. That is $G \setminus (P_1 \cup P_2)$ has at least two components. W.l.o.g. we assume that $R$ is located in the exterior $E = R_i$. 
**Theorem 40:** For any planar graph $G$ we have $c(G) \leq 3$.

Case 1 and Case 2
Theorem 40: For any planar graph $G$ we have $c(G) \leq 3$.

Case 1: Number of neighbors!

$c$ one neighbor in $R_i$: Move all cops to this neighbor $c'$ and Consider $R_{i+1} = R_i \setminus \{c'\}$. Case 1 again.

$c$ more than one neighbor in $R_i$: $a$ and $b$ be two neighbors, $P(a, b)$ a shortest path in $R_i$ between $a$ and $b$. One cop remains in $c$, another cop protects the path $P(a, b)$ by Lemma 39. Thus $P_1 = a, c, b$ and $P_2 = P(a, b)$. Case 2 with $R_{i+1} \subset R_i$. 
**Theorem 40:** For any planar graph $G$ we have $c(G) \leq 3$.

Case 2:
Theorem 40: For any planar graph $G$ we have $c(G) \leq 3$.

Case 2:

1. There is another shortest path $P'(v_1, v_2)$ in $P_1 \cup P_2 \cup R_i$ but different from $P_1$ and $P_2$. Leaves $P_1 \cup P_2$ at $x_1$, hits $P_1 \cup P_2$ again at $x_2$.

2. There is no such path! There is a single vertex $x$ of $P_1 \cup P_2$ so that $R$ is in the component behind $x$. Move all three cops to $x$. Case 1 again!
Shortest path $P'(v_1, v_2)$ in $P_1 \cup P_2 \cup R_i$ but different from $P_1$ and $P_2$. Leaves $P_1 \cup P_2$ at $x_1$, hits $P_1 \cup P_2$ again at $x_2$.

Let $c_3$ protect $P_3 = v_1, \ldots, x_1, r_1, \ldots, r_k, x_2, \ldots, v_2$ while $c_1$ and $c_2$ protect $P_1 \cup P_2$.

Case 2 again: $c_3$ protects $P_3$, $c_1$ or $c_2$ the remaining one!
Aspects of randomization

- Examples for the use of randomizations
- Context of decontaminations
- Randomization for a strategy
- Beat the greedy algorithm for trees
- Randomization as part of the variant
- Probability distribution for the root
- Expected number of vertices saved
Integer LP formulation for trees (Exercise):

Minimize \( \sum_{v \in V} x_v w_v \)

so that \( x_r = 0 = 0 \)

\[ \sum_{v \leq u} x_v \leq 1 : \text{for every leaf } u \]

\[ \sum_{v \in L_i} x_v \leq 1 : \text{for every level } L_i, i \geq 1 \]

\( x_v \in \{0, 1\} : \forall v \in V \)
Strategy: Beat the greedy approximation

- $\text{opt}_{ILP}$ optimal solution, $\text{opt}_{RLP}$ fractional solution, $\text{opt}_{ILP} \leq \text{opt}_{RLP}$
- $\text{opt}_{RLP}$ in polynomial time!
- Subtree $T_v$ with $x_v = a \leq 1$ is $a$-saved, a portion $a \cdot w_v$ of the subtree is saved
- $v_1$ is ancestor of $v_2$ and $x_{v_1} = a_1$ and $x_{v_2} = a_2$
- Vertices of $T_{v_2}$ are $(a_1 + a_2)$-saved. The remaining vertices of $T_{v_1}$ are only $a_1$-saved.
- Randomized rounding scheme for every level
- Sum of the $x_v = a$-values for level $i$: Probability distribution for choosing $v$. Shuffle and set $x_v$ to 1.
- Sum up to less than 1: Probability of not choosing a vertex at level $i$.
- Only problem: double-protections
**Strategy: Beat the greedy approximation**

- *double-protections*: Choose vertices on the same path to a leaf! We only use the predecessor! Skip the higher level!
- No such *double-protections*: The expected approximation value would be indeed 1.
- Intuitive idea: Tree $T_{v_i}$ at level $i$ is *fully* saved by the fractional strategy!
- Worst-case: Fractional strategy has assigned a $1/i$ fraction to all vertices on the path from $r$ to $v_i$. This gives 1 for $T_{v_i}$.
- Probability of saving $v_i$ is: $1 - (1 - 1/i)^i \geq 1 - \frac{1}{e}$.
- Formal general proof!
Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by opt_{RLP}. The expected approximation ratio of the above strategy for the number of vertices protected is \((1 - \frac{1}{e})\).

- \(S_F\) fractional solution for opt_{RLP}
- Probabilistic rounding scheme: \(S_I\) outcome of this assignment
- Show: Expected protection of \(S_I\) is larger than \((1 - \frac{1}{e})\) times the value of \(S_F\)
- \(x_v^F\) value of \(x_v\) for the fractional strategy
- \(x_v^I\) value \(\{0, 1\}\) of integer strategy
- \(y_v = \sum_{u \leq v} x_u \in \{0, 1\}\) indicate whether \(v\) is finally saved
- \(y_v^F = \sum_{u \leq v} x_u^F \leq 1\) fraction of \(v\) saved by fractional strategy
Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by \( \text{opt}_{RLP} \). The expected approximation ratio of the above strategy for the number of vertices protected is \( (1 - \frac{1}{e}) \).

For \( y_v = 1 \) it suffices that one of the predecessor of \( v \) was chosen. Let \( r = v_0, v_1, v_2, \ldots, v_k = v \) be the path from \( r \) to \( v \)

\[
\Pr[y_v = 1] = 1 - \prod_{i=1}^{k}(1 - x_{v_i}^F).
\]

Explanation: The probability that \( v_2 \) is safe is

\[
x_1 + (1 - x_1)x_2 = 1 - (1 - x_1)(1 - x_2)
\]

The probability that \( v_3 \) is safe is

\[
1 - (1 - x_1)(1 - x_2) + (1 - x_1)(1 - x_2)x_3 = 1 - (1 - x_1)(1 - x_2)(1 - x_3)
\]

and so on.
Approximation by randomized strategy

**Theorem 41:** Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\text{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

$$\Pr[y_v = 1] = 1 - \prod_{i=1}^{k} (1 - x_{v_i}^F)$$

$$\geq 1 - \left(\sum_{i=1}^{k} (1 - x_{v_i}^F) \right)^k = 1 - \left( \frac{k - \sum_{i=1}^{k} x_{v_i}^F}{k} \right)^k$$

$$= 1 - \left( \frac{k - y_v^F}{k} \right)^k$$

$$= 1 - \left( 1 - \frac{y_v^F}{k} \right)^k \geq 1 - e^{-y_v^F} \geq \left( 1 - \frac{1}{e} \right) y_v^F.$$
Approximation by randomized strategy

**Theorem 41:** Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\text{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

$$E(|S_I|) = \sum_{v \in V} \text{Pr}[y_v = 1] \geq \left(1 - \frac{1}{e}\right) \sum_{v \in V} y_v^F = \left(1 - \frac{1}{e}\right) |S_F|.$$
Randomization in variants of the problem

- $G = (V, E)$ fixed number $k$ of agents
- $k$-surviving rate, $s_k(G)$, is the expectation of the proportion of vertices saved
- Any vertex is root vertex with the same probability
- Classes, $C$, of graphs $G$: For constant $\epsilon$, $s_k(G) \geq \epsilon$
- Given $G$, $k$, $v \in V$ let:
  - $s_{n_k}(G, v)$: number of vertices that can be protected by $k$ agents, if the fire starts at $v$
  - $\frac{1}{|V|} \sum_{v \in V} s_{n_k}(G, v) \geq \epsilon |V|
- Class $C$: let the minimum number $k$ that guarantees $s_k(G) > \epsilon$ for any $G \in C$ be denoted as the firefighter-number, $ffn(C)$, of $C$. 