Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Dynamic strategies on Trees

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November 3rd, 2015
Corollary 14: Computing a strategy for a tree $T$ of size $n$ that saves at least $k$ vertices can be done in $O(n2^k k)$ time.

- Run above algorithm for $i = 1, \ldots, k$
- Sufficient!
- $\sum_{i=1}^{k} i2^i n \leq kn \sum_{i=1}^{k} 2^i = (2^{k+1} - 2)kn$
Subexponential bound:

Bound for $k$: Show $k \leq \sqrt{2n}$

**Lemma 15:** If a vertex at depth $d$ is burning in an optimal strategy for an instance of the firefighter problem on trees, at least $\frac{1}{2}(d^2 + d)$ vertices are safe.

Proof:

- Optimal strategy, vertex $v$ at depth $d$ burning
- Guard at $v_i$ in every depth $1, 2, \ldots, d$
- $T_{v_i}$ has size $\geq d - i + 1$
- $\sum_{i=1}^{d} (d - i + 1) = \frac{1}{2}(d^2 + d)$
Subexponential bound:

Bound for $k$: Show $k \leq \sqrt{2n}$

**Theorem 16:** There is an $O\left(2^{\sqrt{2n} n^{3/2}}\right)$ algorithm for the firefighter problem on a tree of size $n$.

**Proof:**

- Run the algorithm for $k \leq \sqrt{2n}$: $(n \cdot 2^k \cdot k)$
- Above Lemma: Burning vertex at depth $\sqrt{2n}$, then $n + \sqrt{n/2} > n$ vertices safe? Contradiction!
- All vertices of depth $k = \sqrt{2n}$ has to be safe for an optimal strategy
- Suffices to use this bound!
Dynamic guards in Trees

- Stationary guards vs. dynamic guards!
- NP-hard for general graphs $\Rightarrow$ Trees
- Many different variants: Here *Clearing of edges!*
- Weights for the Corridors. Weights for the vertices.
- Recontamination, if weight is to small!
- Intruder has maximum speed.
Contiguous search strategy

Weighted Graphs $G = (V, E)$

1. Place $p$ guards on a vertex.
2. Move $r$ guards along an edge.

The set of all cleared edges $E_i$ after step $i$ has to be connected!

- Edge weights $w(e)$, vertex weights $w(v)$ with $w(v) \geq w(e)$ for any $e = (v, u) \in E$
- Recontamination by non-protected paths
- Infinite speed for the Intruder
- Example: Blackboard
Optimal contiguous search strategy

Weighted Tree $T = (V, E)$, search number $cs(T)$!

$(\emptyset, \{e_6\}, \{e_6, e_5\}, \{e_6, e_5, e_4\}, \{e_6, e_5, e_4, e_1\}, \{e_6, e_5, e_4, e_1, e_2\}, \{e_6, e_5, e_4, e_1, e_2, e_3\})$ 10 agents required! $cs(T) = 10!$
**Theorem 17:** For any weighted tree $T$ there is a monotone contiguous search strategy with $cs(T)$ agents where all agents initially start at the same vertex $b$. 
• $X \subseteq E$

• *boundary vertices* $\delta(X)$:
  Vertices that have vertices incident to $X$ and $E \setminus X$

• $w(X_i) := \sum_{v \in \delta(X_i)} w(v)$

• $w(\{e_4, e_5, e_6\}) = 7$ and $w(\{e_2\}) = 10$. 

![Graph Diagram]

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Theoretical Aspects of Intruder Search
Optimal contiguous search strategy, Crusade definition

- \((X_0, X_1, \ldots, X_m)\) subsets \(X_i \subseteq E\)
- \(X_0 = \emptyset\) and \(X_m = E\)
- \(|X_i \setminus X_{i-1}| \leq 1\) for \(1 \leq i \leq m\)
- Connected if \(X_i\) connected for \(1 \leq i \leq m\)
- Frontier: \(\max_{1 \leq i \leq m} w(X_i)\)
- Progressive: \(X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m\) and \(|X_i \setminus X_{i-1}| = 1\) for \(1 \leq i \leq m\)
Contiguous search and connected crusade

- \( cs(T) \leq k \) and a contiguous search!
- \( X_i \) set after each step!
- Search step, at most one additional edge, means
  \[ |X_i \setminus X_{i-1}| \leq 1 \]
- \( X_i \) not destructed, means \( w(X_i) \leq k \).
- \( X_i \) connected, because contiguous search
- \( X_0 = \emptyset \) and \( X_m = E \)

**Lemma 18:** For \( cs(T) \leq k \) there is a connected crusade of frontier at most \( k \).
Optimal contiguous search strategy

Weighted Tree $T = (V, E)$, search number $cs(T)$!

$cs(T) = 10!$

(∅, {e_6}, {e_6, e_5}, {e_6, e_5, e_4}, {e_6, e_5, e_4, e_1}, {e_6, e_5, e_4, e_1, e_2},
{e_6, e_5, e_4, e_1, e_2, e_3}) 10 agents required! $cs(T) = 10!$
**Lemma 19:** For $cs(T) \leq k$ there is a *progressive* connected crusade of frontier at most $k$.

Connected crusades $C = (X_0, X_1, \ldots, X_m)$ of frontier at most $k$

Choose one with:

1. $\sum_{i=0}^{m}(w(X_i) + 1)$ is minimum.
2. Among all crusade satisfying condition 1. choose one with: $\sum_{i=0}^{m}|X_i|$ is minimum.

Has to exist, show that this is progressive:

$|X_i \setminus X_{i-1}| = 1$ for $1 \leq i \leq m$
Difference: Connected Crusade, Progessive Crusade

Connected crusade:
\[(\emptyset, \{e_1\}, \{e_1, e_2\}, \{e_2\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}, \{e_3\}, \{e_1, e_3\}, \{e_1, e_2, e_3\}, \{e_3, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_3, e_4\})\]: \(|X_i \setminus X_{i-1}| \leq 1\) for \(1 \leq i \leq m\)

Progressive con. crusade:
\[(\emptyset, \{e_1\}, \{e_1, e_2\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_3, e_4\})\], \(|X_i \setminus X_{i-1}| = 1\) for \(1 \leq i \leq m\), \(X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m\)
1. $\sum_{i=0}^{m}(w(X_i) + 1)$ is minimum.

2. Among all crusade satisfying condition 1. choose one with:
   $\sum_{i=0}^{m}|X_i|$ is minimum.

- Assume: $C = (X_0, X_1, \ldots, X_m)$ with $|X_i \setminus X_{i-1}| = 0$
- Take: $C' = (X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m)$, Condition 1.
- This means $X_i \subseteq X_{i-1}$.
- $|X_{i+1} \setminus X_{i-1}| \leq 1$ from $|X_{i+1} \setminus X_i| \leq 1$ and $X_i \subseteq X_{i-1}$,
- Connected!
- Can assume: $|X_i \setminus X_{i-1}| = 1$ for $1 \leq i \leq m$!
Progressive connected crusade, frontier at most $k$

1. $\sum_{i=0}^{m}(w(X_i) + 1)$ is minimum.
2. Among all crusade satisfying condition 1. choose one with:
   $\sum_{i=0}^{m}|X_i|$ is minimum.

- Prove $X_i \subseteq X_{i-1}$!
- Case 1.: $w(X_{i-1} \cup X_i) < w(X_i)$
- $C' = (X_0, \ldots, X_{i-1}, X_{i-1} \cup X_i, X_{i+1}, \ldots, X_m)$, Cond. 1.!
- $X_i$ and $X_{i-1}$ connected, $X_{i-1} \cup X_i$ is connected since
  $|X_i \setminus X_{i-1}| = 1$
- $|X_{i+1} \setminus (X_{i-1} \cup X_i)| \leq 1$ since $|X_{i+1} \setminus X_i| = 1$.
  If $|X_{i+1} \setminus (X_{i-1} \cup X_i)| = 0$ go back to former case!
- Case 2.: $w(X_{i-1} \cup X_i) \geq w(X_i)$
Progressive connected crusade, frontier at most \( k \)

1. \( \sum_{i=0}^{m} (w(X_i) + 1) \) is minimum.

2. Among all crusade satisfying condition 1. choose one with:
\[ \sum_{i=0}^{m} |X_i| \] is minimum.

- Prove \( X_i \subseteq X_{i-1} ! \)
- Case 2.: \( w(X_{i-1} \cup X_i) \geq w(X_i) \)
- Exercise: \( w(A \cup B) + w(A \cap B) \leq w(A) + w(B) \) link sets \( A, B \)
- \( w(X_{i-1} \cap X_i) \leq w(X_i) \) for \( 1 \leq i \leq m \)
- \( C'' = (X_0, \ldots, X_{i-2}, X_{i-1} \cap X_i, X_{i+1}, \ldots, X_m) \)
- Cond. 2.: \( |X_{i-1} \cap X_i| \geq |X_{i-1}| \) which gives \( X_{i-1} \subseteq X_i \)
- \( |X_i \setminus (X_i \cap X_{i-1})| = |X_i \setminus X_{i-1}| = 1 \) and
\[ |(X_i \cap X_{i-1}) \setminus X_{i-2}| \leq |X_{i-1} \setminus X_{i-2}| \leq 1 \]
- Show that \( C'' \) is connected!!
Progressive connected crusade, frontier at most \( k \)

- \( C'' = (X_0, \ldots, X_{i-2}, X_{i-1} \cap X_i, X_{i+1}, \ldots, X_m) \) connected?
- Ass. \( X_{i-1} \cap X_i \) not connected!
- \( \{e\} = X_i \setminus X_{i-1} \) and \( W = X_{i-1} \setminus X_i \) and \( Z = X_{i-1} \cap X_i \). By assumption \( Z = Z' \cup Z'' \) where \( Z' \) and \( Z'' \) do not share a vertex.
- Contrad. \( T \) is a tree, \( X_{i-1} \cap X_i \) is also connected.
**Lemma 19:** For $cs(T) \leq k$ there is a *progressive* connected crusade with frontier at most $k$.

- Build strategy from *progressive* connected crusade frontier at most $k$!
- First, double the edges $T$, $T'$!
Lemma 20: Let $T'$ be a tree so that every link has at least one vertex of degree 2. If there is a progressive connected crusade of frontier $\leq k$ in $T'$, there is a monotone contiguous search strategy using $\leq k$ guards in $T'$ and the guards can be initially placed at a single vertex $v_1$.

Proof: Inductive argument!

- pcc. $C = (X_0, X_1, \ldots, X_m)$ frontier $\leq k$
- $e_i = (v_i, u_i) := X_i \setminus X_{i-1}$, this order
- Start with $k$ guards at $v_i$
- $w(X_1) = w(v_1) + w(u_1) \leq k$, $w(e_1) \leq w(u_1)$
- move $w(u_1)$ searchers along $w_1$
Lemma 20: $T'$ (every link has vertex of degree 2) and progressive connected crusade of frontier $\leq k$. Monotone contiguous strategy with the same bound!

Proof:

- $e_1, \ldots, e_{i-1}$ without recontaminations
- $e_i = (v_i, u_i)$ incident to $X_{i-1}$, $v_i \in \delta(X_{i-1})$
- Case 1: $w(X_{i-1}) + w(u_i) \leq k$:
  - Clear link $e_i$ by $w(u_i)$ agents move from $v_i$ to $u_i$.
- Case 2: $w(X_{i-1}) + w(u_i) > k$
- Not both vertices $v_i, u_i$ in $\delta(X_i)$
- $v_i \in \delta(X_{i-1})$. Assume $v_i \in \delta(X_i)$
- $\deg(v_i) > 2$ and $\deg(u_i) = 2$
- $u_i \in \delta(X_i)$ implies link $f_i \neq e_i$ containing $u_i$ has to be contaminated and $u_i \notin \delta(X_{i-1})$
- $w(X_i) = w(X_{i-1}) + w(u_i)$ Contradiction!
Case 2: $w(X_{i-1}) + w(u_i) > k$ and not both vertices $v_i, u_i$ in $\delta(X_i)$

$$w(v_i) + k - w(X_{i-1}) \geq w(u_i)$$

3. $w(X_i) = w(X_{i-1}) - w(v_i) + w(u_i)$ and at least $w(v_i)$ guards at $v_i$.

Move all $k - w(X_{i-1})$ free guards to $v_i$.

$$w(v_i) + k - w(X_{i-1}) \geq w(v_i) + w(X_i) - w(X_{i-1}) \geq w(u_i)$$
Lemma 21: Any contiguous monotone strategy for $T'$ can be translated to a contiguous monotone strategy for $T$ with the same number $k$ of agents.

Proof:
Let $e' = (x, y)$ and $e'' = (y, z)$ links stemming extension $e$.

If $q$ guards move from $x$ to $y$ or $z$ to $y$, they stay in place in $T$.

If $q$ guards move from $y$ to $x$ or from $y$ to $z$, they move from $z$ to $x$ or from $x$ to $z$ in $T$, respectively.
**Lemma 22:** Any contiguous monotone strategy for $T$ with $k$ agents can be translated to a contiguous monotone strategy for $T'$ with the same number $k$ of agents.

**Proof:**
A move along an edge $e = (u, v)$ in $T$ is splitted into two moves along $e'$ and $e''$.

$u$ is kept safe: If the move clears $e = (u, v)$, then $q \geq w(e)$ have traversed $e$.

From the construction $q$ searchers are also enough for $w(e) = w(e') = w(e'')$ and the weight $w(e)$ of the intermediate vertex.
Theorem 17: For any weighted tree $T$ there is a monotone contiguous search strategy with $\text{cs}(T)$ agents where all agents initially start at the same vertex $b$.

- $\text{cs}(T') \leq \text{cs}(T)$ (Theorem 22)
- Connected crusade of frontier $\text{cs}(T')$ in $T'$ (Lemma 18)
- Monotone contiguous strategy for $\text{cs}(T')$ in $T'$ with start vertex $b$ (Lemma 20)
- Monotone contiguous strategy for $\text{cs}(T') = \text{cs}(T)$ in $T$ with start vertex $b$ (Lemma 21)
Design of a strategy: Example!

Startvertex \( v \) and order of the subtrees:

\[
\text{cs}(T_v(z)) = \max\{\text{cs}(T_v(z_1)), \text{cs}(T_v(z_2)) + w(z)\}
\]