Theoretical Aspects of Intruder Search
Course Wintersemester 2015/16
Introduction

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Organisation

- Lecture: Tuesday 16:15 to 17:45
- Exercise groups: Starting next week 28/29th
  Wednesday: 14-16
  Thursday: 10-12
- Sign in
- Manuscript on the webpage
- Slides on the webpage
- Exercises
- Today: Introduction, different topics
Main problems and intention

- Evader/Intruder versus Searcher/Guard
- Escaping/Intruding versus Catching/Avoidance
- Game, Competition
- Different Scenarios: Environment, Facilities, Goal, Model
- Discrete, Continuous, Geometry, Combinatorics
- Interpretation: Possible Position of the Intruder, Decontamination, Firefighting
Theoretical Aspects

- Algorithmic track
- Computational complexity
- Correctness or Failure
- Efficiency
- Optimality
- Prerequisites: Algorithms, Datastructure, Analysis, Complexity, Computability
- Models, Methods, Proof Techniques, Tools
- Today Introduction
Example I: Polygon, Safe an Area, Complexity

- Continuous Problem
- Complexity Result
- NP-hardness
- Reduction

**Optimal-Closing-Sequence:**

**Instance:** Simple polygon, set of \(n\) intruders, set of \(m\) doors to be closed successively time \(c_i\), safes area \(A_i\).

**Output:** Compute the optimal sequence of doors that has to be closed for maximizing the area safed.
Reduction: Subset-Sum with threshold

*Subset-Sum:*

**Instance:** $n$ integer numbers $a_1, a_2, \ldots, a_n$, integer threshold $t$

**Output:** Sum of a subset of $a_1, a_2, \ldots, a_n$ as close as possible to $t$, not exceeding $t$.

- Reduction to *Optimal-Closing-Sequence*
- Construct Instance in polynomial time
- Solution for *Optimal-Closing-Sequence* $\iff$ Solution for *Subset-Sum*
Circle radius $r$, center $s$, intruder start at $s$

Chords of length $a_i$, polygonal chain: $A_i, B_i, C_i', D_i$

Door $d_i$ safes Area $A_i = \frac{h a_i}{4}$

Speed $v(t + 0.5) = r$ for every Intruder

Choose $r$ so that $vt < x_i = \sqrt{r^2 - \left(\frac{a_i}{2}\right)^2}$

Substituting $v$ by $\frac{r}{(t+0.5)}$:

$$\left(\frac{a_i}{2}\right)^2 < \left(1 - \frac{t^2}{(t + 0.5)^2}\right) r^2$$
Reach $C_i'$ after $t + 0.5$ steps, do not reach $B_i$ after $t$ steps. Maximize!
Theorem 1: Computing an optimal-enclosure-sequence is NP-hard.

Proof: Reduction from Subset-Sum, Equivalence!
Example II: Grid Graph

- Discrete Problem
- Correctness/Failure
- Structural Properties

Evader-Enclosure in Grid-Graphs

**Instance:** A rectangular grid, a start vertex $s$ of the evader and $k$ protecting guards per time step.

**Output:** Compute an efficient protection strategy that encloses the evader (and finally find the evader).

A Two Player Game!
Example II: Grid Graph, $k = 2$

Evader moves (4Neighborship), Guards will be placed

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Theoretical Aspects of Intruder Search
Example II: Grid Graph, $k = 2$

Evader moves (4Neighborship), Guards will be placed.

- Grid graph with $k = 2$ guards placed at strategic positions to enclose the evader at point $s$. The graph illustrates the concept of intruder search with theoretical aspects.

- The diagram shows a grid with 4 neighbors for each point, highlighting the movement and placement strategy for guards to efficiently enclose the evader.
Example II: Grid Graph, $k = 2$

Evader moves (4Neighborship), Guards will be placed

![Diagram of a grid graph with evader locations marked as $s$, and guards marked as $k = 2$.]
Example II: Grid Graph, $k = 2$

Evader moves (4Neighborship), Guards will be placed

Example Applet! Enclosing the Evader first!
Lemma 2: Catching an evader in a grid world by setting $k = 1$ blocking cells after each movement of the evader cannot succeed in general.

Step I: $r_i$ blocked cells in $D_{i+1}, D_{i+2}, \ldots$
$B_i \subseteq D_i$ burning cells in $D_i$
**Lemma 2:** Catching an evader in a grid world by setting $k = 1$ blocking cells after each movement of the evader cannot succeed in general.

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- Move of the evader: \( B'_{l+1} = 1 + r_l - x + 1 \)
- Block of the guard in \( D_{l_1}: l_1 \leq l + 1 \)
  \[ \Rightarrow r_{l+1} = r_l - x, \ B_{l+1} \geq 1 + r_{l+1} \]
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- Move of the evader: $B'_{l+1} = 1 + r_l - x + 1$
- Block of the guard in $D_{l_1}$: $l_1 \leq l + 1$
  $\Rightarrow r_{l_1} = r_l - x$, $B_{l+1} \geq 1 + r_{l+1}$
- Block of the guard in $D_{l_1}$: $l_1 > l + 1$
  $\Rightarrow r_{l_1} = r_l - x + 1$, $B_{l+1} \geq 1 + r_{l+1}$
Lemma 3: For $k = 2$ there is a successful enclosement strategy, that encloses the evader after 8 steps. After 9 additional steps, the evader will be found.
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Firefigthing interpretation! Outside the fire!

**Lemma 3:** For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successful strategy encloses an area of at least 18 burning vertices. This bound is tight.
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- \( L = \{(x, y)| |x| \leq l \text{ and } |y| \leq l\} \) and \( 0 \leq t \leq T \)
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- \( L = \{(x, y) | |x| \leq l \text{ and } |y| \leq l \} \text{ and } 0 \leq t \leq T \)
- \( b_{v,t} = \begin{cases} 
1 & \text{vertex } v \in L \text{ burns before or at time } t \\
0 & \text{otherwise}
\end{cases} \)
Example II: Grid Graph, $k = 2$

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- $b_{v,t} = \begin{cases} 1 & \text{ vertex } v \in L \text{ burns before or at time } t \\ 0 & \text{ otherwise} \end{cases}$
- $d_{v,t} = \begin{cases} 1 & \text{ vertex } v \in L \text{ is defended before or at time } t \\ 0 & \text{ otherwise} \end{cases}$
Example II: Grid Graph, \( k = 2 \)

Firefigthing interpretation! Integer LP for \( l \leq 8, \ T \leq 9 \)

\[
\begin{align*}
\text{Min} \quad & \sum_{v \in L} b_{v,T} \\
 b_{v,t} + d_{v,t} - b_{w,t-1} & \geq 0 \quad : \ \forall v \in L, v \in N(w), 1 \leq t \leq T \\
 b_{v,t} + d_{v,t} & \leq 1 \quad : \ \forall v \in L, 1 \leq t \leq T \\
 b_{v,t} - b_{v,t-1} & \geq 0 \quad : \ \forall v \in L, 1 \leq t \leq T \\
 d_{v,t} - d_{v,t-1} & \geq 0 \quad : \ \forall v \in L, 1 \leq t \leq T \\
 \sum_{v \in L} (d_{v,t} - d_{v,t-1}) & \geq 2 \quad : \ \forall 1 \leq t \leq T \\
 b_{v,0} & = 1 \quad : \ v \in L \text{ is the origin } (0,0) \\
 b_{v,0} & = 0 \quad : \ v \in L \text{ is not the origin } (0,0) \\
 d_{v,0} & = 0 \quad : \ \forall v \in L \\
 d_{v,t}, b_{v,t} & \in \{0,1\} \quad : \ \forall v \in L, 1 \leq t \leq T
\end{align*}
\]
Example II: Grid Graph, $k = 2$

**Lemma 4:** For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successful strategy encloses an area of at least 18 burning vertices. This bound is tight.

Optimal solution by LP solver:
Example II: Grid Graph, $k = 2$

**Lemma 4:** For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successful strategy encloses an area of at least 18 burning vertices. This bound is tight.

Optimal solution by LP solver:
Example III: Continuous Firefigthing

Geometric Firefigther Problem

Instance: A circle with center $C$ of radius $A$ that grows with unit speed. An agent who builds a firebreak with speed $v > 1$

Output: Compute a strategy that finally fully enclose the spreading fire.
Example III: Continuous Firefigthing

**Geometric Firefigther Problem**

**Instance:** A circle with center $C$ of radius $A$ that grows with unit speed. An agent who builds a firebreak with speed $v > 1$

**Output:** Compute a strategy that finally fully enclose the spreading fire.

A circular strategy!
Lemma 5: Enclosing a fire of extension $A$ with a single circular loop around the source of the fire is possible, if and only if the speed $v$ of the firefighter is larger than $2\pi$.

Proof:
**Example III: Continuous Firefigthing**

**Lemma 5:** Enclosing a fire of extension $A$ with a single circular loop around the source of the fire is possible, if and only if the speed $v$ of the firefigther is larger than $2\pi$.

Proof:

- Choose $p = (A + x, 0)$ away from the fire
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Proof:
- Choose $p = (A + x, 0)$ away from the fire
- Loop around origin: $\frac{2\pi(A+x)}{v}$ time
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Proof:

- Choose $p = (A + x, 0)$ away from the fire
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- Circle expands $\frac{2\pi(A+x)}{v}$, smaller than $x$?
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- Loop around origin: $\frac{2\pi(A+x)}{\nu}$ time
- Circle expands $\frac{2\pi(A+x)}{\nu}$, smaller than $x$?
- Equivalent to $\frac{2\pi A}{x} + 2\pi \leq \nu$
- If and only if $\nu > 2\pi$
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- Choose $p = (A + x, 0)$ away from the fire
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GeoGebra Simulation
Example IV: Firefigthing Grid-World Simulation

*Discrete Firefigther Problem*

**Instance:** Grid contamination of size $B$, spreads 4Neighborship after $n$ time steps. Agent cleans a cell, builds a wall cell and leaves the cell within $b$ time steps.

**Output:** Compute a strategy that finally fully enclose the spreading fire.

Example: $n = 30$, $b = 5$, $B = 3 \times 3$
Example IV: Firefigthing Grid-World Simulation

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*Discrete Firefigther Problem*

**Instance:** Grid contamination of size $B$, spreads 4Neighborship after $n$ time steps. Agent cleans a cell, builds a wall cell and leaves the cell within $b$ time steps.

**Output:** Compute a strategy that finally fully enclose the spreading fire.
Conjecture 1: For a grid fire that spreads after $n$ time steps and an agent that builds a wall within $b$ time steps, the spiral strategy only succeeds if $b < \frac{n-1}{2}$ holds.

By simulation!