Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16

Example Queries for the oral exams

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General remarks!

- Repetition of the main statements: Problem Def./Theorem/Lemmata
- Top-Down! Proof ideas and details!
- Explanation on examples! Algorithm/Lower Bound!
- Example Questions!
- Not all details are on the foils!
- First questions Q1/Q2 in detail!
- Walk-Through!
Graphs and Trees

- Model: Grid environment, static variant, moving agent
- Q: How many agents are required?
  - Q1: Lower bound, proof in detail
  - Q2: Upper bound, proof idea
Lemma 2: Catching an evader in a grid world by setting \( k = 1 \) blocking cells after each movement of the evader cannot succeed in general.

Step I: \( r_I \) blocked cells in \( D_{I+1}, D_{I+2}, \ldots \)

\[ B_I \subseteq D_I \text{ burning cells in } D_I \]
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\begin{itemize}
    \item $s$
    \item $D_1$
    \item $D_4$
    \item $2$
\end{itemize}

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Q1: Proof detail, Lower Bound $k = 1$

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- Block of the guard in $D_{l_1}$: $l_1 \leq l + 1$
  \[ \Rightarrow r_{l+1} = r_l - x, \ B_{l+1} \geq 1 + r_{l+1} \]
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- Block of the guard in $D_{l_1}$: $l_1 \leq l + 1$  
  $\Rightarrow r_{l+1} = r_l - x$, $B_{l+1} \geq 1 + r_{l+1}$
- Block of the guard in $D_{l_1}$: $l_1 > l + 1$  
  $\Rightarrow r_{l+1} = r_l - x + 1$, $B_{l+1} \geq 1 + r_{l+1}$
Lemma 3: For $k = 2$ there is a successful enclosement strategy, that encloses the evader after 8 steps. After 9 additional steps, the evader will be found.
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Firefigthing interpretation! Outside the fire!

Lemma 3: For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successful strategy encloses an area of at least 18 burning vertices. This bound is tight.
Q2: Proof idea, Upper bound! $k = 2$

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- $L = \{(x, y) | |x| \leq l \text{ and } |y| \leq l\}$ and $0 \leq t \leq T$
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- $b_{v,t} = \begin{cases} 
1 & : \text{ vertex } v \in L \text{ burns before or at time } t \\
0 & : \text{ otherwise} 
\end{cases}$
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\]

- \( b_{v,t} = \begin{cases} 
1 : & \text{vertex } v \in L \text{ burns before or at time } t \\
0 : & \text{otherwise}
\end{cases} \)

- \( d_{v,t} = \begin{cases} 
1 : & \text{vertex } v \in L \text{ is defended before or at time } t \\
0 : & \text{otherwise}
\end{cases} \)
Q2: Proof idea, Upper bound! $k = 2$

Firefigthing interpretation! Integer LP for $l \leq 8$, $T \leq 9$

$$\text{Min } \sum_{v \in L} b_{v,T}$$

$$b_{v,t} + d_{v,t} - b_{w,t-1} \geq 0 \quad : \quad \forall v \in L, v \in N(w), 1 \leq t \leq T$$

$$b_{v,t} + d_{v,t} \leq 1 \quad : \quad \forall v \in L, 1 \leq t \leq T$$

$$b_{v,t} - b_{v,t-1} \geq 0 \quad : \quad \forall v \in L, 1 \leq t \leq T$$

$$d_{v,t} - d_{v,t-1} \geq 0 \quad : \quad \forall v \in L, 1 \leq t \leq T$$

$$\sum_{v \in L} (d_{v,t} - d_{v,t-1}) \geq 2 \quad : \quad \forall 1 \leq t \leq T$$

$$b_{v,0} = 1 \quad : \quad v \in L \text{ is the origin } (0,0)$$

$$b_{v,0} = 0 \quad : \quad v \in L \text{ is not the origin } (0,0)$$

$$d_{v,0} = 0 \quad : \quad \forall v \in L$$

$$d_{v,t}, b_{v,t} \in \{0, 1\} \quad : \quad \forall v \in L, 1 \leq t \leq T$$
• Same Model: static variant, moving agent, general graph
• How many agents are required?
• Q: Complexity of the problem?
• Q3: Explain the NP-hardness, present reduction in detail
• Q: Polynomial time in some cases?
  • Q: Special graphs?
  • Q: Greedy approximation for trees: Factor and proof!
  • Q: Dynamic programming approach for trees! Explain!

$k$-Clique and $k' = k + \left( \frac{k}{2} \right) + 1$ protected vertices
- Static: Approximation Greedy, Dynamic Programming (exact)

\[ |T_v| + A_{l(v)}((1, 0, 0), 1) \text{ or } A_{l(v)}((1, 1, 0), 0) \]
Dynamic configuration, structure!

1. Place a team of $p$ guards on a vertex.
2. Move a team of $m$ guards along an edge.
(3. Remove a team of $q$ guards from a vertex)

- Contiguous search (1.+2.) number: $cs(T) \leq k$
- Theorem 17: Monotone contiguous strategy with all $cs(T)$ agents that starts in a single vertex.
- Corollary 33: Tree $T$ exists with $cs(T) \leq 2s(T) - 2$.
- Q5 Definition: Progr. connected crusades, frontier at most $k$
- Q6 Proof idea: Progr. Conn. Crusades frontier $k$, $T$ and $T'$
- Q7 Rule 3. What is the difference? Jumping! $cs(T)$ vs. $s(T)$
Dynamic configuration, trees, strategy

- Message sending algorithm! Q8 Explain the idea! Analysis!
- Correct only for unit weights! Q9 Explain the problem!

\[
\mu(v_3) = \max(\lambda_{v_3}(e_1), \lambda_{v_3}(e_3) + 7) = 12
\]
\[
\mu(v_5) = \max(\lambda_{v_5}(e_4), \lambda_{v_5}(e_5) + 5) = 10
\]
\[
8. \lambda_{v_3}(e_1) = 7
\]
\[
4. \lambda_{v_4}(e_1) = 10
\]
\[
6. \lambda_{v_4}(e_4) = 6
\]
\[
7. \lambda_{v_5}(e_4) = 10
\]
\[
7. \lambda_{v_5}(e_5) = 10
\]
\[
5. \lambda_{v_5}(e_6) = 10
\]
\[
3. \lambda_{v_5}(e_6) = 1
\]
\[
10. \lambda_{v_7}(e_6) = 10
\]
\[
2. \lambda_{v_3}(e_3) = 5
\]
\[
1. \lambda_{v_3}(e_2) = 3
\]
\[
11. \lambda_{v_2}(e_3) = 10
\]
\[
12. \lambda_{v_1}(e_2) = 12
\]
Cop and Robber Problems

- Structural properties: Pitfalls, Classification, If-and-only-if!
- Q10 Explain the concepts/definitions!
- Number of cops required! $c(G)$
- Theorem 41: $G$ max. degree 3, any two adjacent edges are contained in a cycle of length at most 5: $c(G) \leq 3$.
- Theorem 43: For planar graphs: $c(G) \leq 3$
- Q11/12: Explain the proof ideas!
Randomization: Tree, static!

- Greedy approximation: $\frac{1}{2}$, Expected: $1 - \frac{1}{e}$
- Q13: Explain the idea, sketch the analysis!

Minimize $\sum_{v \in V} x_v w_v$
so that $x_r = 0 = 0$

$$\sum_{v \leq u} x_v \leq 1 : \text{ for every leaf } u$$
$$\sum_{v \in L_i} x_v \leq 1 : \text{ for every level } L_i, i \geq 1$$

$x_v \in \{0, 1\} : \forall v \in V$

$$\Pr[y_v = 1] = 1 - \prod_{i=1}^{k} (1 - x_{v_i}^F) \geq \left(1 - \frac{1}{e}\right) y_v^F.$$
Randomization: Search number, random fire

- Minimal number $k$ such that proportional part can be safed
- $s_k(G) \geq \epsilon: \frac{1}{|V|} \sum_{v \in V} s_{n_k}(G, v) \geq \epsilon |V|

Q14: Explain the definitions!

Theorem 46: Planar graphs, no 3- and 4-cycle: $s_2(G) \geq 1/22$.

Analysis:
- Let $X_2$ denote the vertices of degree $\leq 2$.
- Let $Y_4$ denote the vertices of degree $\geq 4$.
- Let $X_3$ denote the vertices of degree exactly 3 but with at least one neighbor of degree $\leq 3$.
- Let $Y_3$ denote the vertices of degree exactly 3 but with all neighbors having degree $> 3$ (degree 3 vertices not in $X_3$).

Q15: Explain the analysis:

$$s_2(G) \geq \frac{n-2}{n} \cdot \frac{x_2 + x_3}{x_2 + x_3 + y_3 + y_4} \geq \frac{n-2}{n} \cdot \frac{x_2 + x_3}{21(x_2 + x_3)} = \frac{n-2}{21n}.$$
Theorem 1: Computing optimal-enclosure-sequence: NP-hard. (Q: Present Reduction!)

Global Greedy! Q16: Explain the prerequisites/the idea!

- Sort remaining jobs $b_j$ by $\frac{A_j(J_n)}{d_j}$, process largest!

1. $b_j$ can be scheduled somewhere in $J_n$. Insert $b_j$: $J_{n+1}$
2. $b_j$ cannot be processed, overlaps with jobs in $J_n$. Find sequence in $J_n$ that overlaps:
   1. Profits of these jobs smaller than $\mu$ times $A_j(J_n)$.
   2. $b_j$ can be scheduled after deletion of the jobs.
   Then build $J_{n+1}$ with $b_j$. Deleted jobs vanish forever!
3. No such sequence exists in $J_n$. Reject $b_j$!
Q16 Explain the analysis in detail!

\[ |J_{\text{opt}}| \leq J_m(\text{blue}) + J_m(\text{green}) + J_m(\text{grey}) \] (1)

\[ \leq \left( 2 + \frac{2}{\mu} \right) (J_m(\text{green}) + J_m(\text{grey})) \] (2)

\[ \leq \frac{2(\mu + 1)}{\mu} (J_m(\text{green}) + \frac{\mu}{1 - \mu} J_m(\text{green})) \] (3)

\[ \leq \frac{2(\mu + 1)}{\mu} \frac{1}{1 - \mu} J_m(\text{green}) \] (4)

\[ \leq 2 \frac{\mu + 1}{\mu(1 - \mu)} J_m(\text{green}) \leq 2 \frac{\mu + 1}{\mu(1 - \mu)} |J_m|. \] (5)

**Explain Inequalities: Grey vs. Green!** (3)

**Paying scheme: Blue vs. Grey and Green** (2)
Theorem 55: Geometric firefighter problem inside a simple polygon with non-intersecting barriers, approximation algorithms saves at least \( \frac{1}{6+4\sqrt{2}} = \frac{3}{2} - \sqrt{2} \approx 0.086 \) times area of the optimal solution.

Q17 Example/Problem with intersecting barriers! Explain!
Spiral strategy is reasonable!

Q18 Explain the relationship: Speed/Spiral!

\[ 1 + \cos(\alpha) \cdot t \]

\[ A + \cos(\alpha) \cdot t \]

\[ |Bq| + \cos(\alpha) \cdot |S^p_q| = |Bp| \]

\[ 1 + \cos(\alpha) \cdot (t + |S^p_q|) \]
Theorem 56: Speed $v > v_I \approx 2.6145$ success of spiralling strat.

Q19 Explain the Strategy Idea!
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\[ \frac{1}{\cos \beta_1} > v_f \]

Speed
Proof of lower speed bound: suppose $v \leq 1.618$.

Theorem 58: Successful spiralling strategy must be of speed $v > \frac{1+\sqrt{5}}{2} \approx 1.618$.

Q19 Explain the Lower Bound construction in detail!

By induction:
On reaching $p_i$, interval of length $A$ below $p_{i-1}$ is on fire.

(Induction base!)
Proof of lower speed bound: suppose \( v \leq 1.618 \).

Theorem 58: Successful spiralling strategy must be of speed

\[ v > \frac{1 + \sqrt{5}}{2} \approx 1.618. \]

Q19 Explain the Lower Bound constr. in detail!

Inductive Step:

After arriving \( p_{i+1} \),
fire moves at least \( x + A \).
Proof of lower speed bound: suppose $v \leq 1.618$.

Theorem 58: Successful spiralling strategy must be of speed $v > \frac{1+\sqrt{5}}{2} \approx 1.618$.

Q19 Explain the Lower Bound constr. in detail!

Inductive Step:
After arriving $p_{i+1}$ fire moves at least $x + A$.
Proof of lower speed bound: suppose $\nu \leq 1.618$.

Theorem 58: Successf. spiralling strategy must be of speed $\nu > \frac{1 + \sqrt{5}}{2} \approx 1.618$.

Q19 Explain the Lower Bound constr. in detail!

On reaching $p_{i+1}$:
1. $A + \frac{x}{v} \leq p_i \leq x$ and
2. $A + \frac{x}{v} + \frac{y}{v} \leq p_{i+1} \leq y$

$$\Rightarrow \frac{1}{\sqrt{v-1}}x + \frac{1}{v-1}A \leq \frac{y}{v}$$
$$\Rightarrow x + A \leq \frac{y}{v}$$

from $v^2 - v \leq 1$
Theorem 59: Strategy FF contains the fire if $v > v_c \approx 2.6144$.

Q20 Explain the idea and sketch the proof!

- $v = 5.27$ ($\alpha = 1.38$)
- $v = 3.07$ ($\alpha = 1.24$)
- $\log(p_0, p_1)$, $\log(p_1, p_2)$
- Wrapping around $\log(p_1, p_2)$
- Free string: $F_1(l)$:
  - Wrapping around $\log(p_0, p_1)$
  - Wrapping around wrappings!
Q20 Explain the idea and sketch the proof!

FollowFire Drawing backwards tangents!

Free strings $F_j/\phi_j$ parameterized by length of starting spirals!

$F_j: l \in [0, l_1]$

$\phi_j: l \in [l_1, l_2]$

$l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$

$l_2 = \frac{A}{\cos \alpha} (e^{2\pi \cot \alpha} - 1) e^{\alpha \cot \alpha}$

$F_{j+1}(l_1) = \phi_{j+1}(l_1)$

$F_{j+1}(0) = \phi_j(l_2)$
Upper bound: Parameterize the free string (Linkage)

Q20 Explain the idea and sketch the proof!

FollowFire Drawing backwards tangents!

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$$F_{j+1}(l_1) = \phi_{j+1}(l_1)$$
$$F_{j+1}(0) = \phi_j(l_2)$$
Q20 Explain the idea and sketch the proof!
Parameterized by length $l$ of starting spirals!

$L_j(l)$ length of the curve! $F_j(l)$ (and $\phi_j(l)$) length of the free string!

Lemma 60:
$L_{j-1} + F_j = \cos \alpha L_j$

Lemma 61:
$\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$
Theorem 59: FollowFire strategy is successful if \( \nu > \nu_c \approx 2.6144 \)

Q21 Explain the meaning of these steps!

When gets the free string to zero?

1. Parameterize free strings for coil \( j \) (Linkage)
2. Structural properties
3. Successive interacting differential equations
4. Inserting end of parameter interval
5. Coefficients of power series
6. Ph. Flajolet: Singularities
7. Pringsheim’s Theorem and Cauchy’s Residue Theorem
Theorem 68: For $\nu > 2$ there successful general strategy. For $\nu \leq 1$ there is no such general strategy. Q22 Present the proofs!
Q23 Give the precise definition
Q24 Explain the proof for Theorem 69-71
Theorem 72: There are examples where a Zig-Zag path is better than the diameter!

Q25 Sketch the construction, give the precise result!

\[ L_1 : Y = \tan \alpha X \]
\[ L_2 : Y = \tan \alpha (b_\alpha - X) \]
\[ (b_\alpha, 0) = V \]
\[ P = (x, 3y) \]
\[ L : Y = 3 \tan \alpha (b_\alpha - X) \]
\[ L_2 : Y = \tan \alpha (b_\alpha - X) \]
\[ d = 3y \]
\[ O = (0, 0) \]
\[ Q = (x, 0) \]
\[ \beta \]
\[ O \quad x \quad Q \quad b_\alpha - x \quad V \]
Q26 Present the idea and the definition! Proof Theorem 73!

Theorem 73: For a set of sorted distances $F_m$ (i.e. $f_1 \geq f_2 \geq \cdots \geq f_m$) we have $\max \text{Trav}(F_m) := \min_i i \cdot f_i$. 
Theorem 74/75: The hyperbolic traversal algorithm solves problem for any list $F_m$ with maximum traversal cost bounded by $D \cdot (\max\text{Trav}(F_m) \ln(\min(m, \max\text{Trav}(F_m))))$ for some constant $D$. There is a lower bound of $d \cdot C \ln \min(C, m)$ with $\max\text{Trav}(F_m(C, A)) \leq C$ for some constant $d$ and arbitrarily large values $C$. Q27 Proof idea!
Alternative cost measure: Certificate path!

Q28 Present the idea and the definition for polygons!
Q29 Explain the extreme cases!

(i) $\alpha_x \approx 2\pi$

(ii) $\alpha_x = 0$
Alternative cost measure: Certificate path!

Sketch the proof for online approximation!

Theorem 76: There is a spiral strategy for any unknown starting point $s$ in any unknown environment $P$ that approximates the certificate for $s$ and $P$ within a ratio of 3.31864.

$$f(\gamma) = \frac{a}{\cos \beta} \cdot e^{\phi \cot \beta} \cdot \frac{e^{\gamma \cot \beta}}{a \cdot e^{(\phi-\gamma)\cot \beta}(1 + \gamma)} = \frac{e^{\gamma \cot \beta}}{\cos \beta(1 + \gamma)} \quad \text{for} \quad \gamma \in [0, 2\pi] \quad (6)$$