Discrete and Computational Geometry

What is discrete geometry?

- Discrete sets: points, lines, circles in $\mathbb{R}^d$
- Structural Properties

I. $n$ lines in the plane

Q: How many regions?

II. $n$ points in the plane

Q: How many of them have the same distance?
What is computational geometry?

Algorithms for solving geometry problems

Convex hull (= a minimum convex polygon that contains a set of points)

Voronoi diagram (all points in a region share the same nearest point site)
Computational Geometry:
Selected Papers

• Chan’s randomized geometric technique
  – Finding the minimum
  – Geometric Dilation

• Voronoi diagrams
  – Abstract Voronoi diagrams
  – Order-$k$ Voronoi diagrams

• Arrangement
  – $k$-level construction
  – Halfspace range reporting

Discrete Geometry:
Jiří Matoušek, Lectures on Discrete Geometry

• Convexity
• Lattice
• Convex Polytope
• Arrangement
• At most $k$-set
• Cutting Lemma and Zone Theorem
Geometry Duality and $k$-sets

2-partition
For two subsets $A$, $B$ of $S$, $A$ and $B$ form a 2-partition of $S$ if $A \cap B = \emptyset$ and $A \cup B = S$.

Given a set $S$ of $n$ points in the plane, how many 2-partitions of $S$ can be separated by a straight line?

General Position Assumption:
No three points of $S$ are on the same line.

How to count such 2-partitions?
1. Consider a straight $L$ not orthogonal to any line $\overrightarrow{pq}$ for any two points $p, q \in S$.
2. Project each point $p \in S$ to $L$ and let $p'$ be the projection point
How to count such 2-partitions? (Continues.)

3. Let \((a_1, a_2, \ldots, a_n)\) be the sequence of projection points on \(L\) (in one direction).

4. A straight line orthogonal to \(L\) and passing between \(a_i\) and \(a_{i+1}\) separates \(S\) into \(i\)-element and \((n - i)\)-element subsets.

5. Consider a point \(c\) on \(L\) and whose \(y\)-coordinate smaller than that of all points of \(S\).

6. Rotate \(L\) at \(c\) counterclockwise.

7. When \(L\) will be orthogonal to \(\overrightarrow{pq}\) for two points, \(p, q \in S\):
   
   - Their projection points are adjacent in the sequence of projection points of \(S\) on \(L\), i.e., if the projection point of \(p\) is \(a_i\), the projection point of \(q\) is \(a_{i+1}\) or \(a_{i-1}\).
   
   - When \(L\) is orthogonal to \(\overrightarrow{pq}\), the two projections are coincident, and after that, their positions in the sequence are swapped.
How to count such 2-partitions? (Continues.)

8. For $1 \leq i \leq n$, let $p_i$ be a point of $S$ whose projection point on $L$ is $a_i$.

9. Before the positions of $a_i$ and $a_{i+1}$ are swapped, \{ $p_1, \ldots, p_{i-1}, p_i$ \} and \{ $p_{i+1}, p_{i+2}, \ldots, p_n$ \} is separated by a straight line orthogonal to $L$ and passing between $a_i$ and $a_{i+1}$.

10. After the positions of $a_i$ and $a_{i+1}$ are swapped, \{ $p_1, \ldots, p_{i-1}, p_{i+1}$ \} and \{ $p_i, p_{i+2}, \ldots, p_n$ \} is separated by a straight line orthogonal to $L$ and passing between $a_i$ and $a_{i+1}$.

# of swaps during the rotation is # of 2-partitions of $S$ which can be separated by a straight line.

$\rightarrow n(n - 1)$.

How to enumerate the $n(n - 1)$ 2-partitions?

An intuitive method

- sort $n(n - 1)/2$ slopes of straight lines passing through two points of $S$
- Following the order of sorted slopes, compute all the $n(n - 1)$ swaps and thus the 2-partitions.
- $O(n^2 \log n)$ time

Can we do better?

- the optimal time is $O(n^2)$
- Using Geometry Duality.
Central Duality $\Psi$

- For a point $p = (a, b) \in \mathbb{R}^2 \setminus \{0\}$, $\Psi(p)$ is a line: $ax + by = 1$.
- For a line $L : ax + by = 1$, $\Psi(L)$ is a point $(a, b)$.

**Fact** For a point $p \in \mathbb{R}^2 \setminus \{0\}$ and a line $L$ not passing through the origin, $p$ and the origin are in the same size of $L$ if and only if $\Psi(L)$ and the origin are in the same side of $\Psi(p)$.

**Lemma**
For a line $L$ not passing through the origin, and a set $S$ of points no of which is the origin, let $S_L$ be the set of points in $S$ which are in the same side of $L$ with the origin, and $S_R$ be the set of points in $S$ which are in the different side of $L$ from the origin.

Then, $\Psi(L)$ and the origin are in the same side of each of $\Psi(S_L)$, but $\Psi(L)$ and the origin are in different sides of each of $\Psi(S_R)$.

**Corollary**
For a point $p \in \mathbb{R}^2 \setminus \{0\}$, and a set $\mathcal{L}$ of lines no of which passes through the origin, let $\mathcal{L}_L$ be the set of lines in $\mathcal{L}$ each of which includes the origin and $p$ in the same side, and $\mathcal{L}_R$ be the set of lines in $\mathcal{L}$ each of which includes the origin and $p$ in the different sides.

Then, $\Psi(p)$ partitions $\Psi(\mathcal{L})$ into $\Psi(\mathcal{L}_L)$ and $\Psi(\mathcal{L}_R)$.
Theorem
Given a set $S$ of $n$ points, it takes $O(n^2)$ time to generate all the $O(n^2)$ 2-partitions of $S$ which can be separated by a straight line.

Sketch of proof

- Assume no of $S$ is the origin; otherwise translate $S$.
- Consider the arrangement $A(\Psi(S))$ formed by the $n$ lines in $\Psi(S)$.
- Due to the central duality, for all points $p$ in a cell of $A(\Psi(S))$, $\Psi(p)$ partition $S$ into the same 2-partition.
- For each two adjacent cells in $A(\Psi(S))$, the corresponding two partitions just differ by one point.
- A depth-first-search can visit all $O(n^2)$ cells of $A(\Psi(S))$ in $O(n^2)$ time.

Definition
Given a set $S$ of $n$ points, a subset $Q$ of $S$ is called a $k$-set if $|Q| = k$ and $Q$ and $S \setminus Q$ can be separated by a straight line. A $\leq k$-set of $S$ is an $i$-set of $S$, $i \leq k$.

Fact
The number of $\leq k$-sets of $S$ is equivalent to the number of switches that occur in the first $k$ positions of the sequence of projection points during the rotation, i.e., the number of switches between $a_i$ and $a_{i+1}$ for $1 \leq i \leq k$.

Theorem
Consider a cyclic sequence of permutations, $P_0, P_1, \ldots, P_{2N} = P_0$, where $N = \binom{n}{2}$, satisfying

1. $P_i$ and $P_{i+N}$ are in reverse order,
2. and $P_{i+1}$ differs from $P_i$ by an adjacent switch.

Then the number of switches in the first $k$ positions for $2N$ consecutive permutations is at most $kn$.

In other words, the number of $\leq k$-sets of $n$ points is at most $kn$. 
Sketch of Proof

• The total number of switches involving an element $b$ is exactly $2n-2$ (twice with any other element).

• If $b$ occurs in a switch in position $i \in (1, 2, \ldots, k)$, it also occurs in a switch in position $n - i$.

• If $i < j < n - i$, by continuity, $b$ occurs in at least two switches in position $j$ (because $b$ will come back to position $i$).

• Any element occurs in at most $2n-2(2-2k-1)=4k$ switches in positions $\{1, 2, \ldots, k\} \cup \{n-k, \ldots, n-1\}$

• The total number of switches in these positions is half of the sum of occurrences of elements in such switches, i.e., $\leq \frac{1}{2}n4k = 2nk$.

• The total number of switches for the first $k$ positions is precisely half of this quantity, i.e., $\leq nk$. 