Discrete and Computational Geometry, WS1516
Exercise Sheet “6”: Order-k Diagrams University of Bonn, Department of Computer Science I

- Written solutions have to be prepared until Wednesday 16th of December, 12:00 pm.
- There is a letterbox in front of Room E.01 in the LBH building.
- You may work in groups of at most two participants.

Exercise 13: Numbers of vertices, edges, and faces of $V_k(S)$ (8 points)

Let $S$ be a set of $n$ point sites in the Euclidean plane satisfying a general position assumption that no three sites are on the same line and no four sites are on the same circle. For $1 \leq i \leq n - 1$, let $N_i$, $E_i$, $I_i$, $S_i$ be the numbers of faces, edges, vertices, and unbounded faces of $V_i(S)$, respectively, and let $S_0$ be 0. Please prove the following:

1. $E_k = 3(N_k - 1) - S_k$ and $I_k = 2(N_k - 1) - S_k$. (Hint: Euler formula. Due the general position assumption, the degree of a Voronoi vertex is 3). (2 points)

2. $N_1 = n$, and $N_2 = 3(n - 1) - S_1$, and $N_k = 3(N_{k-1} - 1) - S_{k-1} - 2 \sum_{i=1}^{k-2}(-1)^{k-2-i}(2(N_i - 1) - S_i)$ implies

$$N_k = 2k(n - k) + k^2 - n + 1 - \sum_{i=0}^{k-1} S_i.$$  

(Hint: By induction on $k$) (6 points)
Exercise 14: Relation between $V_i(S)$ and $V_{i+1}(S)$ (4 points)

Assume $VR_i(H, S)$ has $m$ adjacent regions $VR_i(H_j, S)$, $1 \leq j \leq m$. Let $Q$ be $\bigcup_{1 \leq j \leq m} H_j \setminus H$. Prove that $V_{i+1}(S) \cap VR_i(H, S) = V_1(Q) \cap VR_i(H, S)$. (Hint: prove that for all site $r \in (S \setminus H) \setminus Q$, $VR_1(r, S \setminus H) \cap VR_k(H, S) = \emptyset$. You can first assume the contrary that $\exists r \in (S \setminus H) \setminus Q$, $VR_1(r, S \setminus H) \cap VR_k(H, S) \neq \emptyset$, and then show that it will lead to a contradiction. For any point $x \in VR_1(r, S \setminus H) \cap VR_k(H, S)$, $\tau \tau$ will intersect a Voronoi edge $e$ between $VR_i(H, S)$ and $VR_i(H_j, S)$ for some $j \in \{1, \ldots, m\}$. Let $y$ be the intersection point between $\tau \tau$ and $e$. Discuss nearest neighbors of $y$, which will lead to a contradiction from the viewpoint of $e$ and the viewpoint of $VR_1(r, S \setminus H)$.)

Bonus 3: Voronoi edges of $k^{th}$-order Voronoi diagrams (4 points)

Consider a Voronoi edge $e$ between two adjacent Voronoi regions $VR_k(H_1, S)$ and $VR_k(H_2, S)$, where $S$ is a set of $n$ point sites in the Euclidean plane. Please prove the following.

1. $|H_1 \setminus H_2| = |H_2 \setminus H_1| = 1$

2. The circle centered at a point $x$ in $e$ and touching $p$ and $q$, where $H_1 \setminus H_2 = \{p\}$ and $H_2 \setminus H_1 = \{q\}$, encloses exactly $k - 1$ sites of $S$.

(Hint: Consider $VR_{k-1}(H, S)$ and $V_1(S \setminus H)$, where $e \cap VR_{k-1}(H, S) \neq \emptyset$.)