A convex polytope is a convex hull of finite points in $\mathbb{R}^d$

- bounded convex polyhedron

**Central Geometric Duality $D_0$**

For a point $a \in \mathbb{R}^d \setminus \{0\}$, it assigns the hyperplane

$$D_0(a) = \{ x \in \mathbb{R}^d \mid \langle a, x \rangle = 1 \},$$

and for a hyperplane $h$ not passing through the origin, where $h = \{ x \in \mathbb{R}^d \mid \langle a, x \rangle = 1 \}$, it assigns the points $D_0(h) = a \in \mathbb{R}^d \setminus \{0\}$. 

![Diagram of Central Geometric Duality $D_0$](image)
An interpretation of duality through $\mathbb{R}^{d+1}$

- “Primal” $\mathbb{R}^d$: the hyperplane $\pi = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 1\}$
- “dual” $\mathbb{R}^d$: the hyperplane $\rho = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = -1\}$
- A point $a \in \pi$
  - construct the hyperplane in $\mathbb{R}^{d+1}$ perpendicular to $0a$ and containing 0
  - intersect the hyperplane with $\rho$

$k$-flat is a hyperplane in $(k+1)$ dimensions.
- 0-flat is a point, 1-flat is a line, 2-flat is a plane, and so on.
- The dual of a $k$-flat is a $(d-k-1)$-flat.
Half-space
For a hyperplane $h$ not containing the origin, let $h^-$ stand for the closed half-space bounded by $h$ and containing the origin, while $h^+$ denotes the other closed half-space bounded by $h$. That is, if $h = \{x \in \mathbb{R}^d \mid \langle a, x \rangle = 1\}$, then $h^- = \{x \in \mathbb{R}^d \mid \langle a, x \rangle \leq 1\}$ and $h^+ = \{x \in \mathbb{R}^d \mid \langle a, x \rangle \geq 1\}$.

Duality preserves incidences
For a point $p \in \mathbb{R}^d \setminus 0$ and a hyperplane $h$ not containing the origin,

- $p \in h$ if and only if $D_0(h) \in D_0(p)$.
- $p \in h^-$ if and only if $D_0(h) \in D_0(p)^-$.  
- $p \in h^+$ if and only if $D_0(h) \in D_0(p)^+$.  

Dual set (Polar set)
For a set $X \subseteq \mathbb{R}^d$, the set dual to $X$, denoted by $X^*$, is defined as follows:

$$X^* = \{y \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } x \in X\}.$$  

Illustration for the dual set $X^*$

- Geometrically, $X^*$ is the intersection of all half-spaces of the form $D_0(x)^-$ with $x \in X$.
- In other words, $X^*$ consists of the origin plus all points $y$ such that $X \subseteq D_0(y)^-$.  
- For example, if $X$ is the quadrilateral $a_1a_2a_3a_4$ shown above, the $X^*$ is the quadrilateral $v_1v_2v_3v_4$.  
- $X^*$ is convex and closed and contains the origin.  
- $(X^*)^*$ is the convex hull of $X \cup \{0\}$.
Famous convex polytopes in $\mathbb{R}^3$

**Tetrahedron**
- four triangles
- 6 edges
- 4 vertices

**Octahedron**
- 8 triangles
- 12 edges
- 6 vertices

**Dodecahedron**
- 12 pentagon
- 30 edges
- 20 vertices
Two Types of Convex Polytopes

**H-polyhedron/polytope**
An H-polyhedron is an intersection of finitely many closed half-spaces in $\mathbb{R}^d$. An H-polytope if an bounded H-polyhedron.

**V-polytope**
An V-polytope is the convex hull of a finite point set in $\mathbb{R}^d$

**Theorem**
Each V-polytope is an H-polytope, and each H-polytope is a V-polytope.

**Mathematically Equivalence, Computational Difference**
- Whether a convex polytope is given as a convex hull of a finite point set or as an intersection of half-spaces
- Given a set of $n$ points specifying a V-polytope, how to find its representations as an H-polytope?
- The number of required half-spaces may be astronomically larger than the number $n$ of points

**Another Illustration**
- Consider the maximization of a given linear function over a given polytope.
- For V-polytopes, it suffices to substitute all points of V into the given linear function and select the maximum of the resulting values
- For H-polytopes, it is equivalent to solving the problem of linear programming.

**Dimension** of a convex polyhedron $P$
- Dimension of its affine hull
- Smallest dimension of an Euclidean space containing a congruent copy of $P$
Cubes

• The $d$-dimensional cube as a point set of the Cartesian Product $[-1, 1]^d$
• As a $V$-polytope, the $d$-dimensional cube is the convex hull of the set $\{-1, 1\}^d$ ($2^d$ points).
• As a $H$-polytope, it is described by the inequalities $-1 \leq x_i \leq 1$, $i = 1, 2, \ldots, d$, i.e., by the intersection of $2d$ half-spaces
• $2^d$ points vs. $2d$ half-spaces
• The unit-ball of the maximum norm $\|x\|_\infty = \max_i |x_i|$

![Cubes Diagram]

Crosspolytope

• $V$-polytope: Convex hull of the “coordinates cross,” i.e., the convex hull of $e_1$, $-e_1$, $e_2$, $-e_2$, ..., $e_d$, and $-e_d$, where $e_1, \ldots, e_d$ are vectors of the standard orthonormal basis. For $d = 2$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
• $H$-polytope: Intersection of $2^d$ half-spaces of the form $\langle \sigma, \leq \rangle 1$, where $\sigma$ ranges over all vectors in $\{-1, 1\}^d$.
• $2d$ points vs. $2^d$ half-spaces
• Unit ball of $l_1$-norm $\|x\|_1 = \sum_{i=1}^{d} |x_i|$. 

![Crosspolytope Diagram]
Simplex

A *simplex* is the convex hull of an affinely independent point set in some $\mathbb{R}^d$.

- A $d$-dimensional simplex in $\mathbb{R}^d$ can also be an intersection of $d+1$ half-spaces.
- The polytopes with smallest possible number of vertices (for a given dimension) are simplices.

\[ \begin{align*}
  d = 0 & \quad \quad \quad \quad d = 1 \quad \quad \quad \quad d = 2 \quad \quad \quad \quad d = 3 \\
  (0, 0, 1) & \quad (0, 0, 1) \quad (1, 0, 0) \\
\end{align*} \]

A *regular* $d$-dimensional simplex in $\mathbb{R}^d$ is the convex hull of $d+1$ points with all pairs of points having equal distances.

- Do not have a very nice representation in $\mathbb{R}^d$
- Simplest representation lives one dimension higher
- The convex hull of the $d+1$ vectors $e_1, \ldots, e_{d+1}$ of the standard orthonormal basis in $\mathbb{R}^{d+1}$ is a $d$-dimensional regular simplex with side length $\sqrt{2}$. 
Proof of equivalence of $H$-polytope and $V$-polytope

$\Rightarrow$ (Let $P$ be an $H$-polytope)

- Assume $d \geq 2$ and let $\Gamma$ be a finite collection of closed half-spaces in $\mathbb{R}^d$ such that $P = \bigcap \Gamma$ is nonempty and bounded (By the induction, $(d-1)$ is correct)
- For each $\gamma \in \Gamma$, let $F_\gamma = P \cap \partial \gamma$ be the intersection of $P$ with bounding hyperplane of $\gamma$.
- Each nonempty $F_\gamma$ is an $H$-polytope of dimension of at most $(d-1)$, and it is the convex hull of a finite set $V_\gamma \subset F_\gamma$ (by the inductive hypothesis)
- Claim $P = \text{conv}(V)$, where $V = \bigcup_{\gamma \in \Gamma} V_\gamma$
  - Let $x \in P$ and let $l$ be a line passing through $x$.
  - The intersection $l \cap P$ is a segment, so let $y$ and $z$ be its endpoints
  - There are $\alpha, \beta \in \Gamma$ such that $y \in F_\alpha$ and $z \in F_\beta$
  - We have $y \in \text{conv}(V_\alpha)$ and $z \in \text{conv}(V_\beta)$.
  - $x \in \text{conv}(V_\alpha \cup V_\beta) \subseteq \text{conv}(V)$

$\Leftarrow$ (Let $P$ be a $V$-polytope)

- Let $P = \text{conv}(V)$ with $V$ finite and assume $0$ is an interior point of $P$
- Consider the dual body $P^* = \bigcap_{v \in V} D_0(v)^-$
- Since $P^*$ is an $H$-polytope, $P^*$ is a $V$-polytope (what we just prove)
  - $P^*$ is the convex hull of a finite point set $U$
- Since $P = (P^*)^*$, $P$ is the intersection of finitely many half-spaces
  - $P = \bigcap_{u \in U} D_0(u)^-$