Exercise 28: The FIFO paging algorithm (4 points)

On a page fault, the FIFO algorithm evicts the page that has been in the cache longest.

1. Show that FIFO is not a marking algorithm.

2. Prove that FIFO is a conservative algorithm.

Exercise 29: The $k$-th harmonic number (4 points)

Prove that for any natural number $k \geq 1$

$$\ln k < H_k \leq 1 + \ln k,$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$. 
Exercise 30: The Full Access Cost Model (4 points)

We consider an alternate cost model the paging problem. We charge a cost of 1 for an access to the fast cache (serving a request whose page is already in the cache) and a cost of \( s \geq 1 \) for moving a page into the cache (additional cost for accessing the page are not included in \( s \)).

Let \( ALG \) be any marking algorithm. Given a request sequence \( \sigma \), we denote by \( ALG(\sigma) \) and \( OPT(\sigma) \) the cost of \( ALG \) and the cost of an optimal offline algorithm, respectively for processing \( \sigma \). Both \( OPT \) and \( ALG \) use a cache of size \( k \). Let \( p \) denote the number of \( k \)-phases in \( \sigma \), and \( L(\sigma) = \frac{|\sigma|}{p} \) be the average length of a \( k \)-phase in \( \sigma \). Show that

\[
\frac{ALG(\sigma)}{OPT(\sigma)} \leq 1 + \frac{(k - 1)s}{L(\sigma) + s}
\]

holds for any \( \sigma \). Furthermore conclude that this implies that \( ALG \) is \( \frac{k(s+1)}{k+s} \)-competitive. You may assume \( L(\sigma) \geq k \).