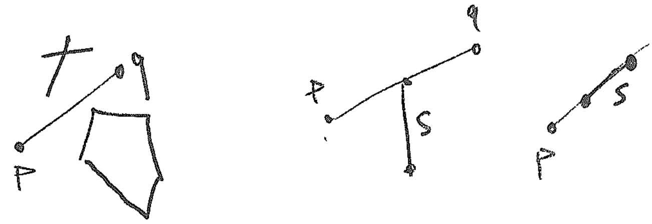
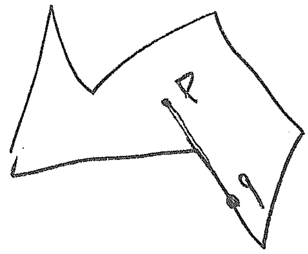
Visibility $s_1, \dots, s_n$ set of line segments in the plane

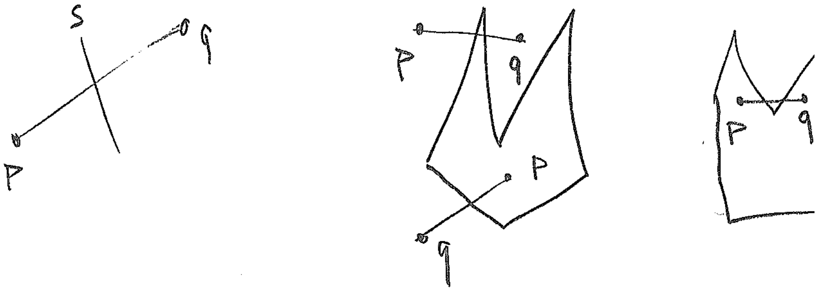
Def: $p, q \in \mathbb{R}^2$ are (mutually) visible $:\Longleftrightarrow$ line segment $\overline{pq}$ not crossed*) by any s_i

Examples p, q visible:





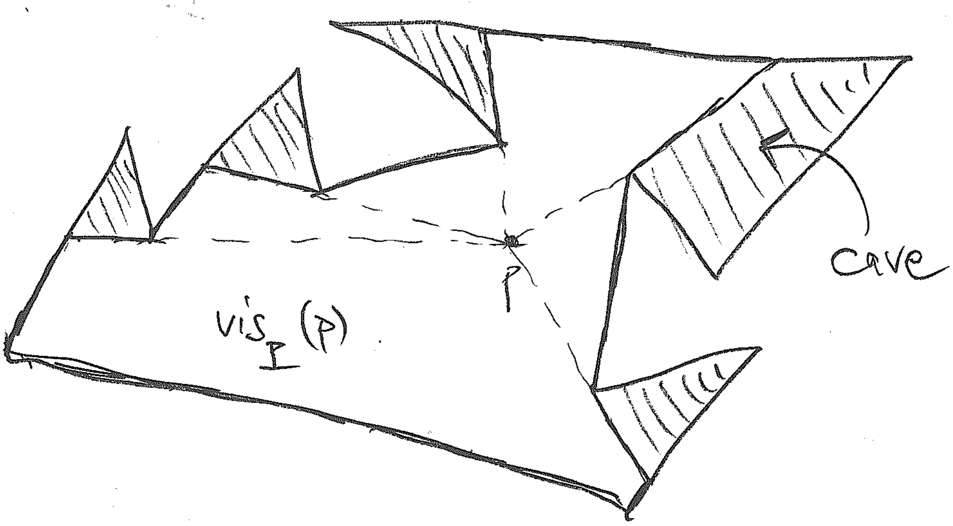
p, q not visible



Special case: segments s_i, $1 \le i \le n$ form simple polygon, $\underline{P}$.

Def: For p inside $\underline{P}$: $vis_{\underline{P}}(p) := \{ q \in \underline{P} \mid p, q \text{ visible} \}$
visibility polygon of p with respect to $\underline{P}$

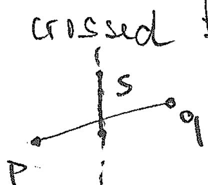
Example



$\underline{P}$

---

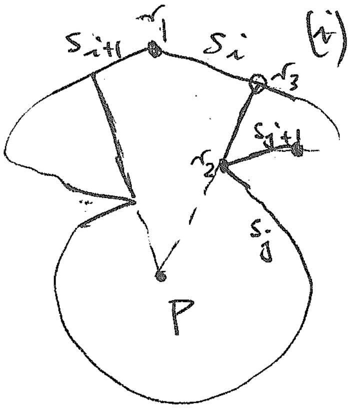
*) $\overline{pq}$ crossed by s $:\Longleftrightarrow$ p, q situated in different open halfplanes in $\mathbb{R}^2 \setminus$ line(s)
and $\overline{pq} \cap s$ = interior point of bot[h]



Lemma: Let $P = (s_1, s_2, \ldots, s_n)$ be a simple polygon with n edges (vertices). Then, for each $p \in P$, vis(p) has $\le n$ v

Proof: vis(p) has 3 types of vertices r

old



(i) both segments s_i, s_{i+1} partially visible from p : r_1

(ii) only one segment partially visible : r_2

new (iii) hit point of ray from p through type (ii) vertex : r_3

Clear: For each new (iii) vertex, one original vertex of P is not visible from p. □

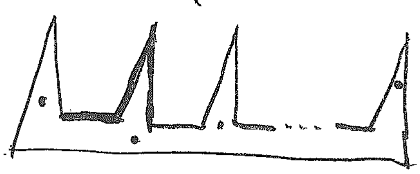
If P convex, all vertices are visible from any $p \in P$.

Classical problem "Guarding an art gallery"

How many $p_1, p_2, \ldots, p_m \in P$ (guards) are needed such that $P \subseteq \bigcup_{i=1}^{m}$ vis

- to determine minimum number m : NP-complete
- $\left\lfloor \frac{n}{3} \right\rfloor$ always sufficient, sometimes necessary.

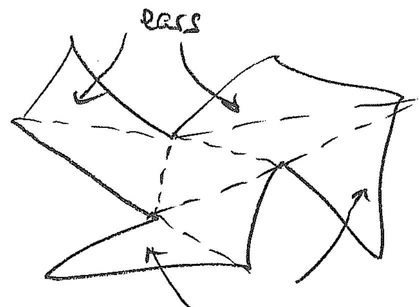
necessary:



needs $\frac{n}{3}$ guards

sufficient: - triangulate P

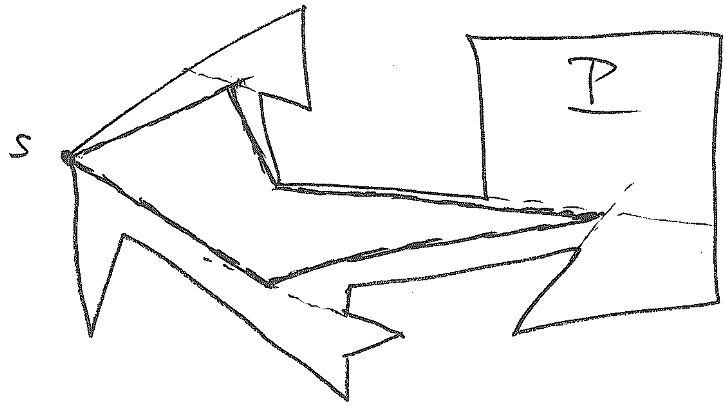
- observe that triangulation contains ≥ 2 ears, i.e. triangles with ≥ 2 edges of P



- use this to prove triangulation to be 3-colorable
- place guards at vertices whose color was used least often.

Interesting to us   mobile guard

walks around to see each $p \in P$ at least once, from gi
start point, s.

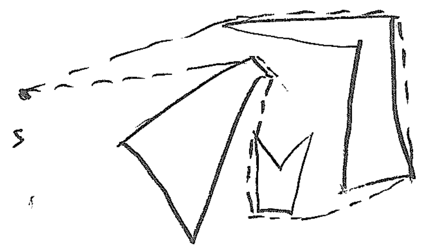


a watchman tour,
computable if P is kn

True or false:

(i) If a watchman has seen, on his tour, all edges of
he has also seen all interior points of P

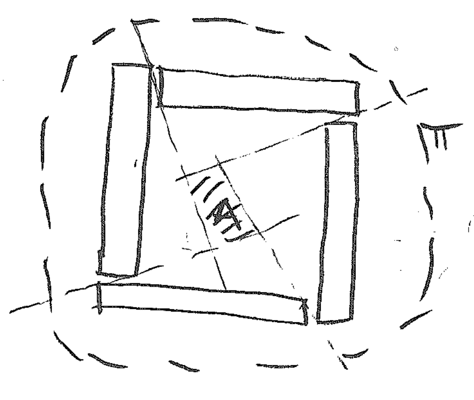
(ii) Same for a watchman guarding a set of obstacles from the outside



Answers

(i): true. Suppose watchman has seen each edge on tour π, but not interior point $p \in P$
$\Rightarrow$ π hidden in cave of $vis_P(p)$
$\Rightarrow$ watchman has not seen edge e ↯

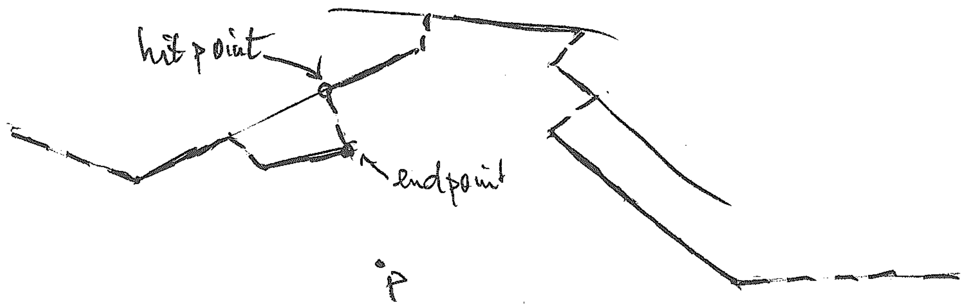
(ii): false



π sees all edges,
but no point of A.

Visibility amidst obstacles

(i) obstacles non-crossing, for example n line segments

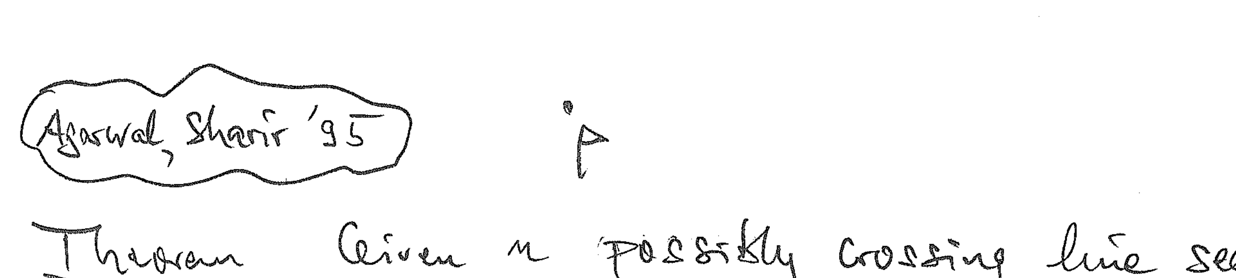


similar to visibility polygon in simple polygon, but not necessarily closed.

Lemma Given n non-crossing line segments, the parts visible from a point have $\leq 4n$ vertices.

Proof n segments have $2n$ endpoints. Each new vertex must be a hit point caused by a ray from an endpoint. An endpoint gives rise to at most one hit point.

(ii) obstacles crossing, for example n line segments



Agarwal, Sharir '95

Theorem Given n possibly crossing line segments, the parts visible from a point have $O(n\alpha(n))$ vertices, and this bound can be attained.

Here, $\alpha(n)$ = "inverse" of Ackermann function

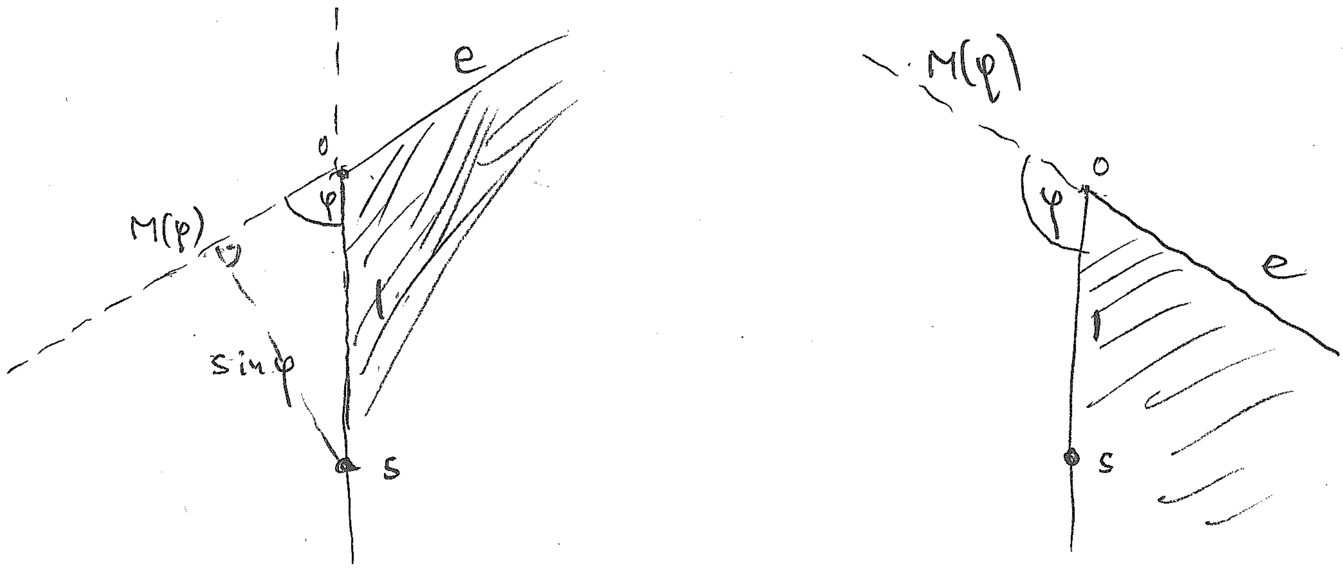
$$A(1,n) = 2n \qquad A(k,1) = A(k-1,1)$$
$$A(k,n) = A(k-1, A(k,n-1))$$

$$A(1,n) = 2n \qquad A(2,n) = 2^n \qquad A(3,n) = \left.2^{2^{2^{\cdot^{\cdot^{2}}}}}\right\} n \text{ times}$$

A grows extremely fast; μ-recursive (programmable with WHILE)
but not primitive recursive (not programmable with only FOR)

$\alpha(m) :=$ smallest n such that $A(n,n) \geq m$
grows extremely slowly.

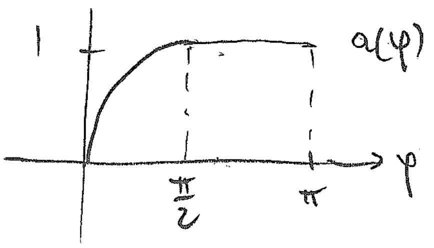
Next online navigation problem:
<u>How to look around a corner</u>



robot wants to "see" edge e, invisible from s
needs to reach extension $M(\varphi)$

optimum solution $a(\varphi)$ = distance from s to $M(\varphi)$

$$= \begin{cases} \sin\varphi, & \varphi \in [0, \frac{\pi}{2}] \\ 1, & \varphi \in [\frac{\pi}{2}, \pi] \end{cases}$$



continuously differentiable

<u>Definition</u> Curve $S = (\varphi, s(\varphi))$ in polar coordinates (defined on $[0, \delta]$, ...) is called strategy for corner problem iff

(i) $s(0) = 1$ (start at point s)

(ii) $\delta < \pi \Rightarrow s(\delta) = 0$ (vertex 0 reached)

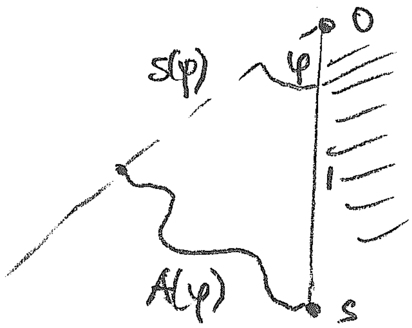
(iii) s continuous on $[0, \delta]$ (no jumps)
piecewise continuously differentiable on $(0, \delta)$
$s'(0) \leq \infty$ exists.

path walked by strategy S up to angle φ:

$$A_S(\varphi) = \int_0^{\varphi} \sqrt{s'^2(t) + s^2(t)}\, dt$$

(arc length in polar coordinates)

s differentiable!



want to measure performance of S dependent on φ:

$$\underbrace{f_S(\varphi)}_{\text{competitive function}} := \frac{A_S(\varphi)}{a(\varphi)}, \quad \text{for all } \varphi \in [0, \delta]$$

path

opt

$f_S(0) := \lim_{0 \leftarrow \varphi} f_S(\varphi)$ if exists

S is called c-competitive $:\Leftrightarrow$

$$\forall \varphi \in [0, \delta] : \quad f_S(\varphi) \le c$$

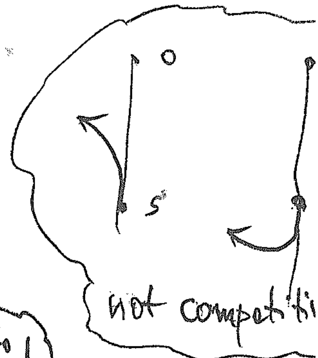
(no additive constant allowed)

then, $c_S := \sup_{\varphi \in [0, \delta]} f_S(\varphi)$ is the competitive factor of S.

Surprisingly, each strategy that "moves away" from $\overline{SO}$ is competitive.

<u>Lemma</u> S competitive $\Leftrightarrow$ $|s'(0)| < \infty$

Then, $c_S \ge \sqrt{s'(0)^2 + 1}$.



<u>Proof</u>

$$c_S \ge f_S(0) = \lim_{0 \leftarrow \varphi} \frac{A_S(\varphi)}{a(\varphi)}$$

$$\underset{\text{l'Hospital}}{=} \lim_{0 \leftarrow \varphi} \frac{A'(\varphi)}{a'(\varphi)} = \frac{\sqrt{s'(\varphi)^2 + s(\varphi)^2}}{\cos(\varphi)} = \sqrt{s'(0)^2 + 1}$$

goes to 1

main theorem diff / int.

goes to 1

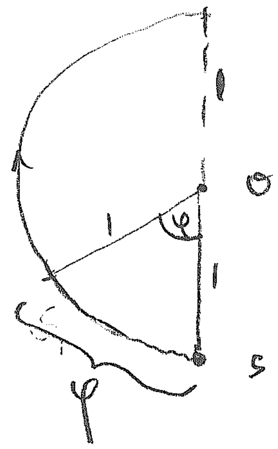
So, $\sqrt{s'(0)^2 + 1} = \infty \Rightarrow c_S = \infty \Rightarrow S$ not competitive

$\sqrt{s'(0)^2 + 1} = f_S(0) < \infty \Rightarrow$ continuous f_S takes on finite maximum on $[0, \delta]$

$\Rightarrow S$ competitive. $\boxed{\text{Lemma}}$

Examples

(i)

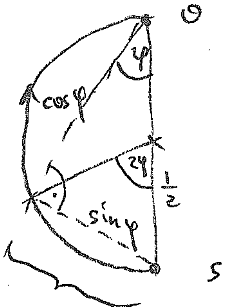


$$f_{S_1}(\varphi) = \begin{cases} \frac{\varphi}{\sin\varphi}, & \varphi \in [0, \frac{\pi}{2}] \\ \varphi, & \varphi \in [\frac{\pi}{2}, \pi] \end{cases}$$

$$\Rightarrow C_{S_1} = f_{S_1}(\pi) = \pi = 3.1414\ldots$$

(ii)

S_2 reaches O at angle $\delta = \frac{\pi}{2}$



$\frac{1}{2}(2\varphi) = \varphi$

(radius!)

$$f_{S_2}(\varphi) = \frac{\varphi}{\sin\varphi}, \quad \varphi \in [0, \frac{\pi}{2}]$$

$$\Rightarrow C_{S_2} = f_{S_2}(\frac{\pi}{2}) = \frac{\frac{\pi}{2}}{1} = \frac{\pi}{2} = 1.5$$

⊕ simple strategy,

⊕ OK performance (?)

⊕ reaches $M(\varphi)$ at exactly the same point as shortest path from s would

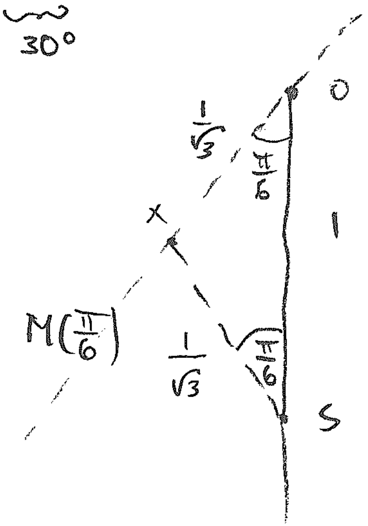
$\Rightarrow$ will apply S_2 later.

Lower bound

Theorem No corner strategy can achieve competitive factor $< \frac{2}{\sqrt{3}} = 1.1547\ldots$

Proof Each strategy must reach $\underbrace{M(\tfrac{\pi}{6})}_{30°}$.

Let $x \in M(\frac{\pi}{6})$, $|xO| := \frac{1}{\sqrt{3}}$



Case 1 robot hits $M(\frac{\pi}{6})$ to the left of x

define $\varphi := \pi$

$\Rightarrow$ robot must still walk to O

$\Rightarrow$ robot's path $\geq |Sx| + |xO| = \frac{2}{\sqrt{3}}$
shortest path $= |SO| = 1$

Case 2 robot hits $M(\frac{\pi}{6})$ to the right of x

define $\varphi := \frac{\pi}{6}$

$\Rightarrow$ robot's path $\geq |Sx| = \frac{1}{\sqrt{3}}$
shortest path $= \sin\frac{\pi}{6} = \frac{1}{2}$ $\boxed{\text{Theorem}}$

That leaves us with a gap $\underbrace{\frac{2}{\sqrt{3}}}_{1.1547..} \ldots \underbrace{\frac{\pi}{2}}_{1.5708..}$

Idea: perhaps optimum strategy R maintains constant competitive function $f_R(\varphi)$?

More wishful thinking: R should arrive at O at angle $\delta \leq \frac{\pi}{2}$

(i) First, we explore what these requirements imply.
(ii) Then, we construct strategy that does fulfill requirem
(iii) Finally, we prove optimality.

(i) We want R to satisfy

$$c = f_R(\varphi) = \frac{A_R(\varphi)}{\sin\varphi} = \frac{\int_0^{\varphi} \sqrt{r'(t)^2 + r(t)^2}\, dt}{\sin\varphi} \quad \forall \varphi \in$$

$$\Rightarrow \quad c \cdot \cos\varphi = (c \cdot \sin\varphi)' = \left(\int_0^{\varphi} \sqrt{r'(t)^2 + r(t)^2}\, dt \right)' = \sqrt{r'(\varphi)^2 + r(\varphi)^2}$$

and $r(0) = 1$ (start position)

$r(\delta) = 0$ (robot reaches θ)

$r(\varphi) > 0$ for $\varphi \in [0, \delta)$.

$$(*) \quad \Rightarrow \quad r'(\varphi) = -\sqrt{c^2 \cos^2\varphi - r^2(\varphi)}$$

radius must be decreasing

let $r(\varphi) = c \cdot u(\varphi)$

$$u'(\varphi) = -\sqrt{\cos^2\varphi - u^2(\varphi)} \qquad (**)$$

Ordinary differential equation

(ii) to find minimum $c > 1$ such that $(**)$ has solution $u(\varphi)$ on $[0, \delta] \subseteq [0, \frac{\pi}{2}]$ satisfying

$$\underbrace{u(0) = \frac{1}{c}}_{\text{initial value}}, \quad u(\varphi) > 0 \text{ for } \varphi \in [0, \delta], \quad u(\delta) = 0$$

(necessary: $u(\varphi) \le \cos(\varphi)$ because of $(**)$)

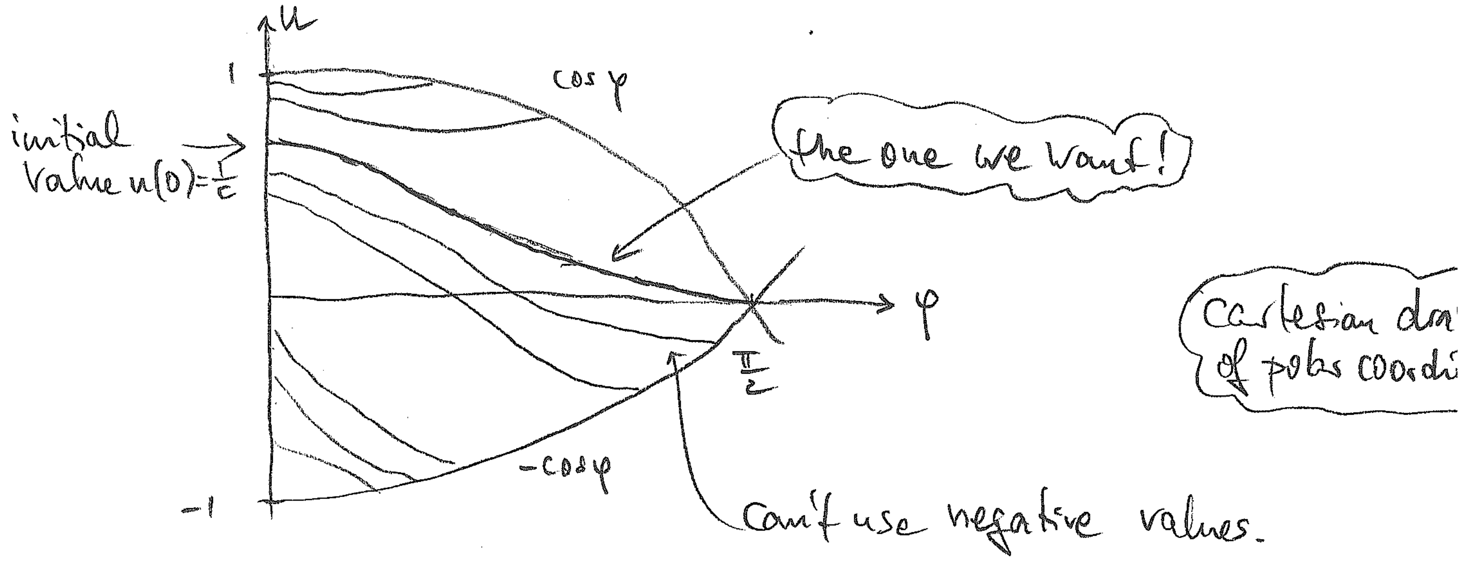
Remark $(**)$ can be transformed into

$$w'(x) = (w^2(x) + 1)(1 - w(x) \cot x)$$ Abelian type;

but apparently closed-form solutions not known

$\Rightarrow$ need to improvise!

Numerical solutions look like this:



Looks promising, but no substitute for a proof. Need to prove that things really are what they seem to be!

Let $D := \{(\varphi, u) \mid 0<\varphi<\frac{\pi}{2},\ |u|<\cos t\}$ open domain

$f(\varphi,u) := -\sqrt{\cos^2\varphi - u^2}$ defined on $\overline{D}$.

A flow in D

continuously differentiable in u

$\Rightarrow$ fulfills local Lipschitz condition in u:

$\forall\ (\varphi_0, u_0) \in D \quad \exists\ N \subset D \quad \exists\ L:$

$(\varphi_0, u_0) \in N, \quad |f(\varphi, u_1) - f(\varphi, u_2)| \leq L \cdot |u$

$\forall\ (\varphi, u_i) \in N$

$\Longrightarrow$ Picard-Lindelöf

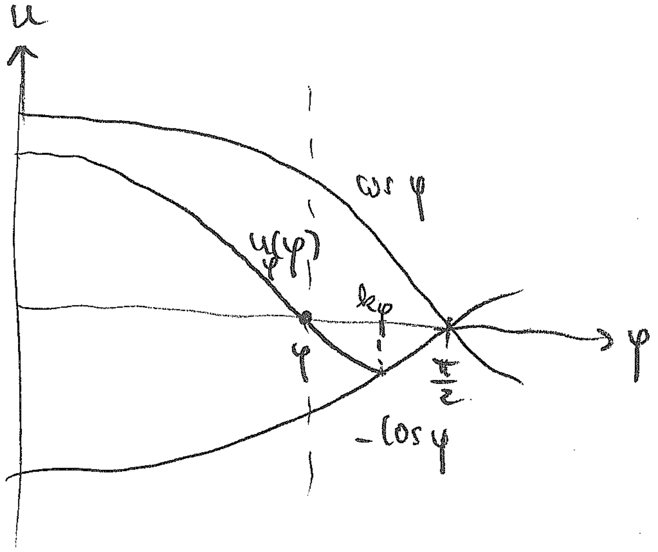
$\forall\ (\varphi_0, u_0) \in D \ \exists$ function $u(\varphi)$ such that

$u(\varphi_0) = u_0, \quad u'(\varphi) = -\sqrt{\cos^2\varphi - u^2(\varphi)}$

$u(\varphi)$ extends to ∂D.

in particular, $\forall \varphi \in (0, \frac{\pi}{2})$ we have such $u = u_\varphi$ satisfying

$u(\varphi) = 0$.



Let $k_\varphi = \max \varphi : u_\varphi(p)$
defined

$\left.\begin{matrix} f<0 \\ (\ast\ast) \end{matrix}\right\} \Rightarrow u_\varphi$ is strictly decreasing $\Rightarrow u_\varphi(k_\varphi) <$

and $(k_\varphi, u_\varphi) \in \partial D \Rightarrow u_\varphi(k_\varphi) = -\cos\varphi.$

to the left of vertical line through φ, u_φ increases.

Cannot hit upper $\cos\varphi$ curve!

Otherwise, there is l such that $u_\varphi(l) = \cos l$

$u_\varphi(t) < \cos t \quad \forall t >$

$$\Rightarrow 0 = \sqrt{\cos^2 l - u_\varphi^2(l)} \underset{(\ast\ast)}{=} u_\varphi'(l) = \lim_{\varepsilon \to 0} \frac{u_\varphi(l+\varepsilon) - u_\varphi(l}{\varepsilon}$$

$$\leq \lim_{\varepsilon \to 0} \frac{\cos(l+\varepsilon) - \cos l}{\varepsilon} = -\sin l < 0 \quad \lightning$$

Hence, u_φ starts at vertical u-axis, i.e., for $\varphi =$

This shows: for each $\varphi < \frac{\pi}{2}$, there does exist solution as shown in numerical picture passing through $(\varphi, 0)$.

With more analysis: Can also find solution for $\varphi = \frac{\pi}{2}$.

Lemma There exists a unique solution $u(\varphi)$ of $u'(\varphi) = -\sqrt{\cos^2\varphi - u^2(\varphi)}$ satisfying $u(\frac{\pi}{2}) = 0$
$u(\varphi) > 0$ for all $\varphi \in [0, \frac{\pi}{2})$. (without proof)

all other don't mak to the co

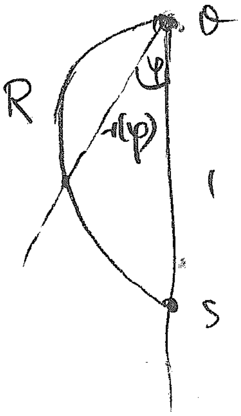
Numerically, $u(0) = \frac{1}{1.21218\ldots} =: c$

$\Rightarrow$ strategy R given by $r(\varphi) = c \cdot u(\varphi)$ is $1.21218..$ competitive.

(pretty close to lower bound $1.1547\ldots$)

(iii) **Theorem** Strategy $R := (\varphi, r(\varphi))$ is optimu
Cornering is of competitive complexity $1.21218\ldots$

Lemma R forms a convex curve



Proof Shows that curvature

$$\kappa = \frac{r^2 + 2r'^2 - rr''}{(r'^2 + r^2)^{3/2}}$$ is positive everywhere.

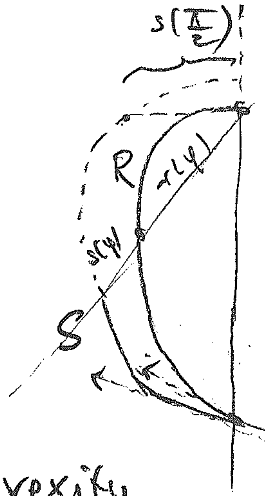
Let $S = (\varphi, s(\varphi))$ be an arbitrary corner strategy.
wlog $|s'(0)| < \infty$ (otherwise S is not competitive)

case 1 $s'(0) \leq r'(0) = -\sqrt{c^2 \cos(0)^2 - r(0)^2} = -\sqrt{c^2 -$

$\Rightarrow c_S \geq \sqrt{s'^2(0) + 1} \geq \sqrt{r'^2(0) + 1} = c$

$\Rightarrow S$ is worse than R.

Case 2 $\quad s'(0) > r'(0)$



$$\Rightarrow \exists \psi : \ s(\varphi) > r(\varphi) \quad \forall \varphi \in [0,\psi]$$

2a) $\quad \forall \varphi \in [0,\pi] : \ s(\varphi) > r(\varphi) : \ \Rightarrow$

$c_S \geq A_S(8) \geq A_S(\frac{\pi}{2}) + s(\frac{\pi}{2}) > A_S(\frac{\pi}{2}) = c$ by convexity

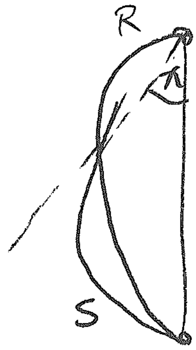
$\Rightarrow$ S is worse.

2b) $\quad \exists \chi \in (0, \frac{\pi}{2}] : \ s(\chi) = r(\chi), \ s(\varphi) > r(\varphi) \ \forall \varphi \in ($

$\Rightarrow A_S(\chi) > A_R(\chi)$ by convexity

$$\Rightarrow c_S \geq f_S(\chi) = \frac{A_S(\chi)}{\sin\chi} > \frac{A_R(\chi)}{\sin\chi} = c$$

here we use that f_R is $\equiv c$!



$\boxed{\text{Theorem}}$

There is no closed form representation for optimum R in polar coordinates; but

$$\alpha = \arcsin \frac{r}{c \cdot \cos\varphi},$$

$c = 1.21218\ldots$