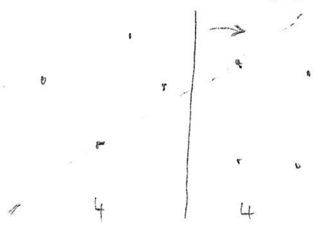
AlgGeo 20.1 **Halving Lines:** Given: $n \equiv 0\ (2)$ points in the plane, general position

Question: How many halving lines exist?



passes through 2 pts

$\frac{n-2}{2}$ pts on either side

Trivial: $O(n^2)$

Theorem: $O(n^{4/3})$

Proof: (Dey)

G planar graph
deg ≥ 3 : $e \le 3v$ (Euler)
no  Ü 1.6 (ii)

Lemma 1: $e > 4v$ $\Rightarrow$ at least $\underbrace{\frac{1}{64}\frac{e^3}{v^2}}_{> v}$ crossings

G simple drawn in $\mathbb{R}^2$

Proof: Consider drawing of G with k crossings

Make this planar by removing $\le k$ edges (1 per $\times$)

$\Longrightarrow$ graph stays simple

$\Longrightarrow$ $e-k \le$ new #edges $\le 3v$, as above (*)

Now H := subgraph of G where each vertex picked with prob. $p := \frac{4v}{e} < 1$

edges survive if both endpoints picked

$\Longrightarrow$ (*) $\quad k_H \ge e_H - 3v_H$

$\Longrightarrow$ E linear $\quad \underbrace{E(k_H)}_{p^4 k} \ge \underbrace{E(e_H)}_{p^2 e} - 3\underbrace{E(v_H)}_{p v}$

$\uparrow$ wlog each crossing involves 4 vertices 

otherwise  reduces # crossings by one

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now,

$$k \geq \frac{e}{p^2} - \frac{3v}{p^3} \underset{p=\frac{4v}{e}}{=} \frac{e^3}{4^2 v^2} - \frac{3\,e^3}{4^3 v^2} = \frac{4e^3-3e^3}{4^3 v^2} = \frac{e^3}{64\,v^2}$$

$\boxed{\text{Lemma 1}}$

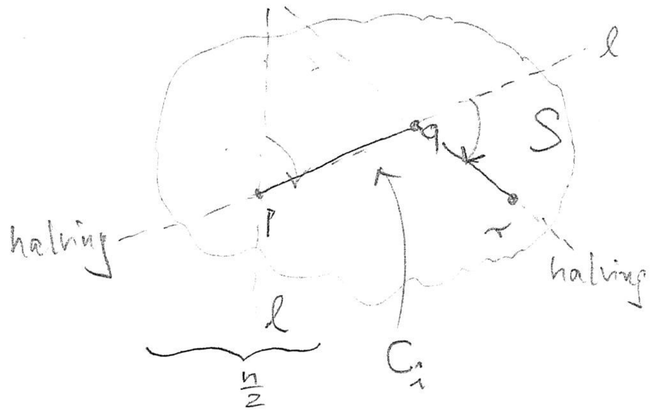
To prove the theorem, let S be set of n points in general position in $\mathbb{R}^2$ (heavily used)

$H := \{ pq ;\ p,q \in S,\ p \text{ left of } q,\ \ell(p,q) \text{ is halving line for } S \}$

Need to show $|H| \in O(n^{\frac{4}{3}})$

Split H into $\frac{n}{2}$ convex chains C_i as follows:

For each point p of the $\frac{n}{2}$ leftmost points do



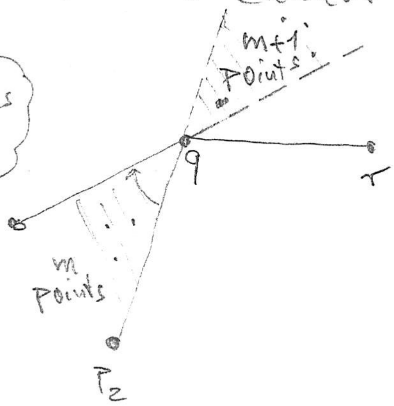
rotate vertical line ℓ clockwise until halving line $\ell(p,q)$ is reached

rotate ℓ clockwise around q until halving line $\ell(q,r)$ is reached and so on until ℓ is vertical again

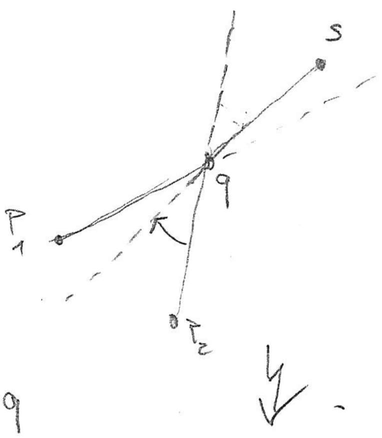
Lemma 2 Each $pq \in H$ occurs in exactly one chain C

Proof (i) Two chains cannot share a segment qr:

as $\ell(p_2,q)$ moves clockwise around q to $\ell(p_1,q)$, right half-plane gains $m+1$ points



because no points lie on dashed lines, by general position



$\Rightarrow$ there exists halving line in between that continues $p_2 q$

$\lightning$.

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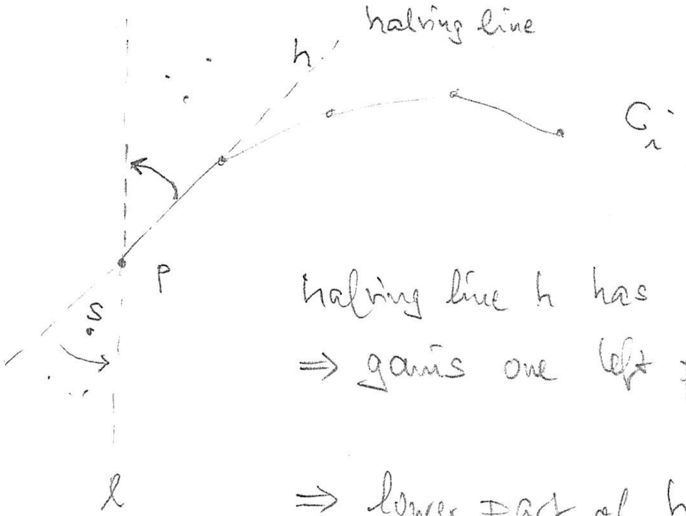
(ii) Each chain C_i has a left endpoint p that belongs to the $\frac{n}{2}$ leftmost points in S



assume $\geq \frac{n}{2}$ points to left of l

halving line h has $\frac{n}{2}-1$ points to its left

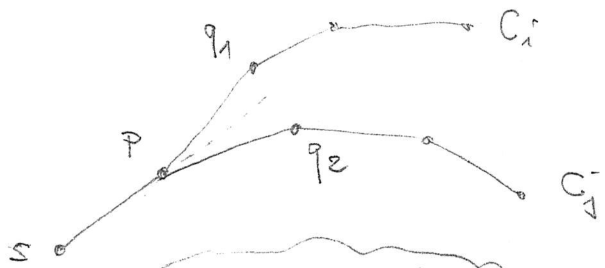
$\Rightarrow$ gains one left point during rotation around p

$\Rightarrow$ lower part of h must hit points like s in lower wedge of h and l

$\Rightarrow$ one of them forms halving line $l(s,p)$

$\Rightarrow$ p is not left endpoint of C_i. ↯

(iii) No two chains have left endpoint p in common



☺!

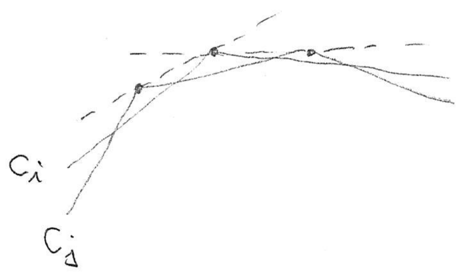
Same argument as before

Otherwise, there exists halving line $l(s,p)$ with slope in between slopes of $l(p,q_1)$ and $l(p,q_2)$ ↯

$\boxed{\text{Lemma 2}}$

This shows: H is split into $\frac{n}{2}$ convex chains that share neither edges nor left endpoints

But two chains can intersect each other



how often?

$\leq$ # common upper tangents

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each line $\ell(v,w)$, $v,w \in S$, can be common upper tangent to at most 2 chains (general position)

$\Rightarrow$ only $O(n^2)$ chain intersections, i.e., intersections between segments in H

Apply Lemma 1 to graph with edges from H over n vertices

$$\frac{1}{64} \frac{e^3}{n^2} \underset{\substack{\uparrow \\ \text{Lemma 1}}}{\leq} \# \text{crossings} \leq C \cdot n^2$$

$\Rightarrow \quad e^3 \leq 64C \cdot n^4$

$\Rightarrow \quad e \in O(n^{\frac{4}{3}})$. $\boxed{\text{Theorem}}$

Tamal Dey, Improved bounds for planar k-sets and related problems, Discrete & Comp. Geom. 19: 373 – 382, 1998

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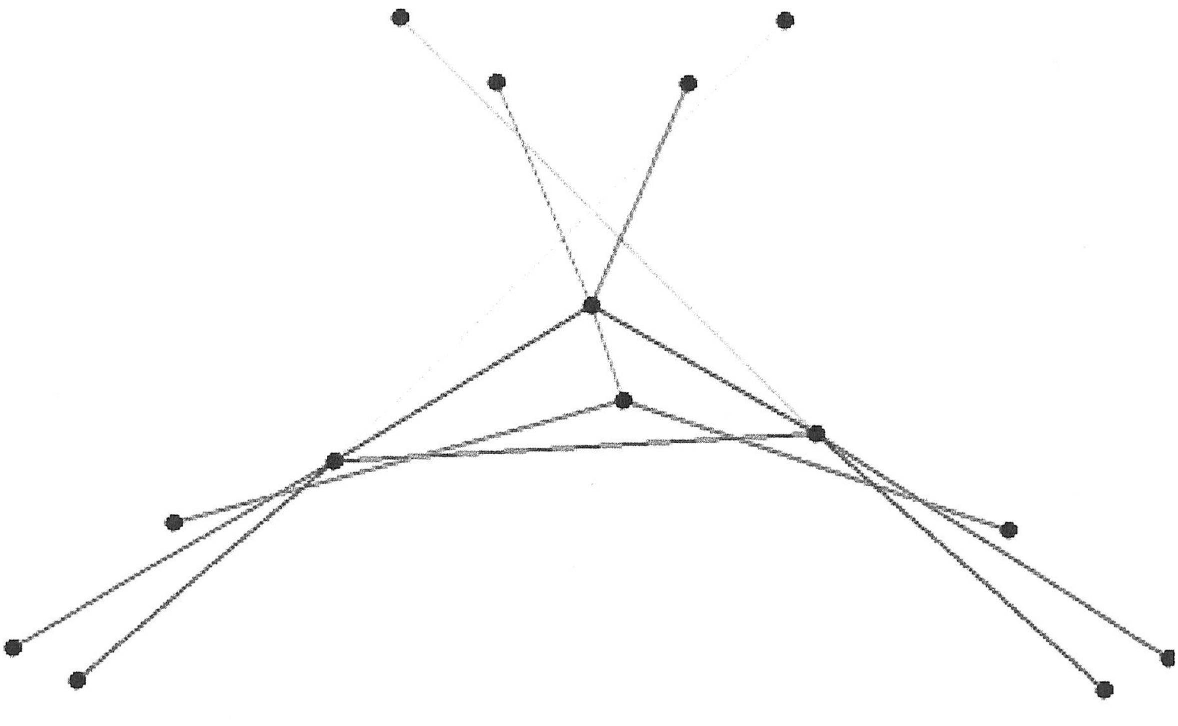
$n = 14$

14 halving lines

7 convex chains



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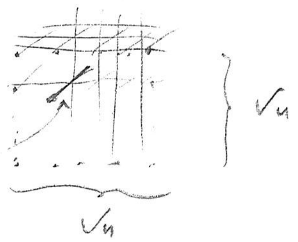
On lines and joints (Kaplan, Sharir, Shustin '18)

Given: n lines in $\mathbb{R}^3$ with
m joints (intersection points of ≥ 3 lines, not coplanar.)

Theorem: $m \in O(n^{\frac{3}{2}})$ tight:



each of these lines carries $\sqrt{n}$ joints

d arbitrary: joint = intersection point of $\geq d$ lines not all in same hyperplane

$m \in O(n^{\frac{d}{d-1}})$

Proof: (i) construct polynomial $p(x_1, \dots, x_d) \neq 0$ of degree $\leq b$ that vanishes on all m joints

(ii) remove all lines carrying $< \frac{m}{2n}$ joints and their joints

(iii) Check that $m > A\, n^{\frac{d}{d-1}}$ implies $\frac{m}{2n} > b$

(iv) Show that a polynomial of degree b that vanishes on $> b$ points per line must be $= 0$ $\lightning$.

(i) Let $b := \left\lfloor \left(d!\,(m+1)\right)^{\frac{1}{d}} \right\rfloor$

$$\Rightarrow (b+1)(b+2)\dots(b+d) \geq d!\,(m+1)$$

$$\Rightarrow \binom{b+d}{d} \geq m+1$$

a general polynomial $p(x_1, \dots, x_d) = \sum\limits_{i=(i_1,\dots,i_d)} a_i x_1^{i_1} \cdots x_d^{i_d}$

of degree $\max\limits_i (i_1 + \dots + i_d) = b$ has $\binom{b+d}{d}$ monomials :

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(ii)

imagine marking d out of $b+d$ positions

$$\underbrace{o\ o\ o}_{i_1}\ \underbrace{\ }_{i_2=0}\!x\ x\ \underbrace{o\ o}_{i_3}\ x\ \underbrace{o}_{i_4}\ x\ o\ o \qquad d=4$$

$\longrightarrow x_1^3 x_3^2 x_4$ has degree $6 \leq 8 = b$.

requiring $p(b_j) = 0$ for $b_j = (b_{j,1}, \dots, b_{j,d})$, $1 \leq j \leq m$ (joints)

means solving a homogeneous system of m linear equations

with $> m$ variables (the coefficients a_i of p)

$\longrightarrow$ there is nontrivial solution.

(ii) at most $n \cdot \frac{m}{2n} = \frac{m}{2}$ joints are removed

each remaining line contains $\geq \frac{m}{2n}$ joints.

(iii) Let $A := (4^d d!)^{\frac{1}{d-1}}$ and assume

$m > A n^{\frac{d}{d-1}} = (4^d d!)^{\frac{1}{d-1}} n^{\frac{d}{d-1}}$

$\Rightarrow m^{d-1} > 4^d d! \, n^d$

$\Rightarrow m^d > 4^d d! \, n^d m$

$\Rightarrow m > 4n (d! \, m)^{\frac{1}{d}}$

$\Rightarrow \frac{m}{2n} > 2 (d! \, m)^{\frac{1}{d}} \geq (d! \, (m+1))^{\frac{1}{d}} \geq b$

(iv) Let $\vec{a}$ be a joint on line $\ell = \vec{a} + t\vec{v}$

let $g_\ell(t) = p(\vec{a} + t\vec{v})$

$\exists \; \frac{m}{2n} > b$ joints on ℓ $\Rightarrow$ g_ℓ has $> b \geq \text{degree}(g_\ell)$ many zeros

$\Rightarrow g_\ell = 0$

Taylor: $\underset{\substack{\| \\ 0}}{g_\ell(t)} = \underset{\substack{\| \\ 0}}{g_\ell(0)} + \underbrace{g_\ell'(0)} \cdot t + O(t^2)$

cannot cancel out

$$g_\ell'(0) = \begin{pmatrix} \frac{\partial p}{\partial x_1}(\vec{a}) \\ \vdots \\ \frac{\partial p}{\partial x_d}(\vec{a}) \end{pmatrix} \cdot (v_1, \dots, v_d) = \nabla p(\vec{a}) \cdot \vec{v}$$

Gradient

$\Rightarrow \quad \Delta p(\vec{a}) \cdot \vec{v} = 0$

same for the other $d-1$ lines passing through $\vec{a}$ joint $\Rightarrow$ $l_1, \dots, l_d$ span d-space

$\Rightarrow \quad \Delta p(\vec{a}) = 0$

same argument for all other joints

$\Rightarrow$ all polynomials $\frac{\partial p}{\partial x_j}$ vanish on all joints

degree $\leq b-1$

by iteration: all higher derivatives of p are $= 0$

contradiction! Because $p \neq 0$ implies that eventually we obtain $\frac{\partial^s p}{\partial x_j \partial x_k \dots \partial x_r} = \text{const} \neq 0$.

$\square$

(Taylor: $f(x) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(x_0)}{\nu!} (x - x_0)^\nu$ if f is analytic

here: $x_0 = 0$, $x = t$)